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Capture Zone Envelopes for Pumping Wells in Fields of Uniform Areal Recharge

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Abstract

The capture zone envelope of a pumping well consists of the streamlines that bound the area of the aquifer that is eventually captured by the well. An approach is outlined to delineate the capture zone envelopes of pumping wells in aquifers with uniform areal recharge. Three types of uniform areal recharge are considered: recharge resulting in straight head contours, circular head contours, and elliptical head contours. Two types of stagnation points are present for wells in uniform areal recharge: saddle points and high points. Each capture zone envelope passes through a saddle point. A new complex polynomial expression is derived of which the stagnation points are among the roots. The streamlines that form the capture zone envelope are traced from saddle stagnation points by numerically integrating the analytic flow field against the direction of flow. The approach may be applied to steady Dupuit-Forchheimer flow (both confined and unconfined) in homogeneous aquifers with a horizontal base. Short Python scripts have been developed and are made available to find all stagnation points, delineate the capture zone envelopes, and visualize the results. Examples are presented for flow fields with multiple wells in a variety of background flows created by uniform areal recharge.

Keywords: Groundwater, Capture Zone Delineation, Analytic Solution, Complex Potential

1 Introduction

The delineation of capture zones is a basic step in the design and protection of water supply wells, and in the design of pump-and-treat systems. A capture zone is commonly delineated

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for a certain time. For example, all the water in a 5-year capture zone reaches the well within 5 years. The capture zone envelope encompasses all the capture zones. As such, the capture zone envelope is the boundary between the water that is (eventually) captured by the well and the water that is not captured by the well.

Every well in a steady, two-dimensional flow field without areal recharge has a stagnation point associated with it [e.g., 3]. The capture zone envelope passes through this stagnation point. The procedure to delineate the capture zone envelopes of a well field commonly consists of two steps: First, find the stagnation points in the flow field, and second determine the streamlines that pass through the stagnation points. Several alternative methods have been developed for these two steps, depending on the complexity of the flow field.

Analytic equations for the stagnation point for one well in a uniform background flow can be found in many standard texts [e.g., 2, 7]. Javandel and Tsang [11] derived equations for the stagnation points of up to three wells with the same discharge, equally spaced on a line perpendicular to a uniform background flow. Shan [21] presented equations for the stagnation points of two arbitrarily located wells in a uniform background flow. Christ and Goltz [4] expanded on the work of [21] and provided equations for the stagnation points of up to four arbitrarily located wells in uniform background flow and gave polynomial equations of which the roots (the stagnation points) can be obtained numerically for systems with more wells. Fienen et al. [6] considered an arbitrary number of wells in uniform background flow and allowed for an anisotropic conductivity field. They determined the stagnation points by finding the roots of the same polynomial equations as [4] by computing the eigenvalues of the companion matrix [e.g., 18]. Similar procedures were applied for wells near a vertical barrier wall [1], a single well between two parallel rivers [10], and multiple wells in wedge-shaped aquifers [20], peninsula-shaped aquifers [27], rectangular domains [27], strip-shaped aquifers [16], and aquifers bounded by open or closed polygons [17]. All these cited references use a complex potential and are formulated such that the stagnation points are the roots of a complex polynomial.

Areal recharge was considered in only a few studies. Finding the stagnation points for flow fields with areal recharge is more challenging, because it cannot be formulated in terms of a complex potential. Lerner [13] derived an equation for the stagnation point for a single well in a strip aquifer with uniform areal recharge. Bakker and Strack [3] developed a numerical procedure to find stagnation points of an arbitrary number of wells in a fairly general background flow field simulated with analytic elements including uniform areal recharge; this approach was implemented in the computer program CZAEM [24]. Mesa and Anderson [15] used stagnation points to trace capture zone envelopes for wells with uniform (radial) recharge and a vertical barrier wall in the shape of a circular arc. Lu et al. [14] presented exact equations for the stagnation point(s) for a single well plus a finite circular recharge area in an otherwise uniform background flow. In this paper, equations are derived for the stagnation points for multiple wells with a background flow consisting of uniform areal recharge and/or uniform flow.

As mentioned, the capture zone envelopes may be delineated once the locations of the stagnation points are known. In the absence of areal recharge, stagnation points are saddle points. Water flows away from a saddle point in two directions and towards a saddle point from two other directions; the three-dimensional potential surface mimics the shape of a saddle around a stagnation point [e.g., 6, Fig. 1] . The two streamlines that form the capture zone envelope end at a stagnation point. One of the streamlines that emanates from a stagnation point ends at the well. The four streamlines that all meet at a stagnation point may be delineated by contouring the stream function [e.g., 2, 4]. This is not done here, however, because that approach only works for the case without areal recharge (i.e., flow governed by Laplace's equation). Another downside of contouring the stream function is that a branch cut, a jump in the stream function, extends from each well, which complicates the delineation of the streamlines. Alternatively, the streamlines that form the capture zone envelope may be computed through numerical integration of the analytic discharge vector against the direction of flow, for example using a standard Runge-Kutta method [e.g., 2, 3, 6]. Two starting points are needed, on opposite sides of each stagnation point, to start these streamlines. These starting points may be chosen somewhat arbitrarily, because the velocity around a stagnation point is very small [3] or more formally using the Hessian matrix of the discharge potential [6].

The objective of this paper is to outline a procedure to delineate the capture zone envelopes of multiple wells in a background flow that is the result of uniform areal recharge. The aquifer is approximated as homogeneous, which is a common approximation in wellhead protection studies [e.g., 12]; the effect of heterogeneous aquifer properties on capture delineation has been investigated by, e.g., [5, 19, 25]. This paper is structured as follows. First, the solution of stagnation points for multiple wells in a uniform background flow without areal recharge is briefly reviewed and it is demonstrated that the stagnation points can be found and the capture zones delineated with only a few lines of Python code using standard packages for scientific computing. Next, new polynomial equations are derived to determine the stagnation points for multiple wells in a variety of flow fields with uniform areal recharge. The delineation of the capture zone envelopes from the stagnation points uses the same procedure as for the case without areal recharge. Several examples are presented to demonstrate the veracity of the presented approach.

2 Wells in Uniform Background Flow

The derivation of equations for the stagnation points of multiple wells in a uniform background flow is reviewed here briefly and it is described how they can be computed using the numpy package of Python [8]. Consider steady Dupuit-Forchheimer flow in a single homogeneous aquifer with a horizontal base; use is made of a Cartesian x, y coordinate system. The discharge vector is defined as the vertically integrated flux in the aquifer and is written as minus the gradient of a discharge potential [e.g., 2, 22]

$$Q_x = -\frac{\partial \Phi}{\partial x}, \qquad Q_y = -\frac{\partial \Phi}{\partial y}, \tag{1}$$

where Φ [L^3/T] is the discharge potential, and Q_x and Q_y [L^2/T] are the x and y components of the discharge vector. For confined flow, or aquifers with a constant saturated thickness H [L], the discharge potential is related to the head h [L] as

$$\Phi = kHh, \tag{2}$$

where k [L/T] is the hydraulic conductivity. For unconfined flow, the relationship is

$$\Phi = \tfrac{1}{2}k(h - z_b)^2, \tag{3}$$

where z_b [L] is the elevation of the bottom of the aquifer.

In the absence of areal recharge, the discharge potential fulfills Laplace's equation

$$\nabla^2 \Phi = 0, \tag{4}$$

and the problem may be formulated in terms of a complex potential [e.g., 2, 22]

$$\Omega = \Phi + \mathrm{i}\Psi, \tag{5}$$

where Ψ is the stream function, and i is the imaginary unit. The complex discharge function $W = Q_x - \mathrm{i}Q_y$, the complex conjugate of the hodograph, may be obtained from the complex potential as

$$W = Q_x - \mathrm{i}Q_y = -\frac{\mathrm{d}\Omega}{\mathrm{d}z}, \tag{6}$$

where $z = x + \mathrm{i}y$ is the complex coordinate.

The complex potential for N wells in an otherwise uniform background flow $W = W_u$ is

$$\Omega = -W_u z + \sum_{n=1}^{N} \frac{Q_n}{2\pi} \ln(z - z_n), \tag{7}$$

where z_n and Q_n [L^3/T] are the complex location and the discharge (positive for pumping water out of the aquifer) of well n, respectively. The corresponding discharge vector is

$$W = W_u - \sum_{n=1}^{N} \frac{Q_n}{2\pi} \frac{1}{z - z_n}. \tag{8}$$

The contributions of all wells may be combined into one fraction

$$\sum_{n=1}^{N} \frac{Q_n}{2\pi} \frac{1}{z - z_n} = \frac{F}{P}, \tag{9}$$

where F is a polynomial in z of order $N - 1$

$$F = \sum_{n=1}^{N} \frac{Q_n}{2\pi} \prod_{\substack{m=1 \\ m \neq n}}^{N} (z - z_m), \tag{10}$$

and P is a polynomial in z of order N

$$P = \prod_{n=1}^{N} (z - z_n). \tag{11}$$

The discharge vector may now be written as

$$W = W_u - \frac{F}{P}. \tag{12}$$

The stagnation points are found by setting W (12) equal to zero, which gives

$$W_u P - F = 0. \tag{13}$$

This is a polynomial in z of order N. The N roots of (13) respresent the N stagnation points in the aquifer.

The polynomial package of the Python package numpy [8] is used to define and manipulate polynomials and to find their roots. It includes the functionality to define a polynomial by specifying the roots, adding and multiplying polynomials, and computing the roots of polynomials. The latter is done by computing the eigenvalues of the companion matrix, as was already applied for groundwater flow by [6]. These functionalities are enough to assemble the polynomial (13) and compute its complex roots (the stagnation points) as shown in Appendix A.

The second step in the delineation of the capture zone envelopes is tracing streamlines from the stagnation points against the direction of flow. The discharge vector field is known analytically (8). There are several Python packages to delineate streamlines from analytic vector fields (e.g., the integration package of scipy [26]). The simplest method is probably the method implemented in the streamplot function of the matplotlib package [9]. The input of this function requires the discharge vector evaluated on a regular grid plus starting locations of the streamlines. The streamplot function then computes the streamlines using a second-order Runge-Kutta method and linear interpolation of the discharge vector between grid points. The streamlines representing the capture zone envelope are obtained by specifying the integration direction as backward, i.e., against the flow. The streamlines cannot start exactly at the stagnation point, of course, as the discharge vector equals zero there. Two starting points are chosen a small distance ε from a stagnation point and on opposite sides. The distance ε is chosen so small that it is indiscernible that the streamlines don't start from the same point. For visual interpretation of the results, two additional streamlines are computed, starting from the same two starting locations but integrated forward, i.e, with the flow. One of these streamlines ends at the pumping well (provided the well is extracting water) and the other flows away from the stagnation point.

3 Example: Capture Zone Envelopes for Five Wells in Uniform Flow

As an example, consider five pumping wells in an otherwise uniform background flow $Q_x = 0.4$ m^2/d and $Q_y = 0.3$ m^2/d. The locations and pumping rates of the wells (Case 1) are listed in Table 1. The hydraulic conductivity and aquifer thickness (confined) or aquifer bottom elevation (unconfined) are not needed as the capture zones depend on the discharge function only. The capture zones are delineated using the approach outlined in the previous. The results are shown in Figure 1a. Note that the capture zone of well 2 consists of four parts: one part North of well 3, one part between wells 3 and 1, one part between wells 1 and 5, and one part to the East of well 5.

For the second case, wells 3 and 5 are injecting water rather than extracting water (see Table 1). The corresponding capture zone envelopes are shown in Figure 1b. Part of the water injected by well 3 now flows to well 2, while part of the water injected by well 5 flows to well 1 and another part flows to well 2.

			Case 1	Case 2
well	x_w (m)	y_w (m)	Q (m^3/d)	Q (m^3/d)
1	-75	0	100	100
2	50	50	100	100
3	-50	100	50	-50
4	-150	-25	150	150
5	0	-100	100	-100

Table 1: Well data used in examples.

4 Flow Solutions for Uniform Areal Recharge

For the case of uniform areal recharge at a rate N_r [L/T] (positive for infiltration), the discharge potential fulfills Poisson's equation

$$\nabla^2\Phi = -N_r. \tag{14}$$

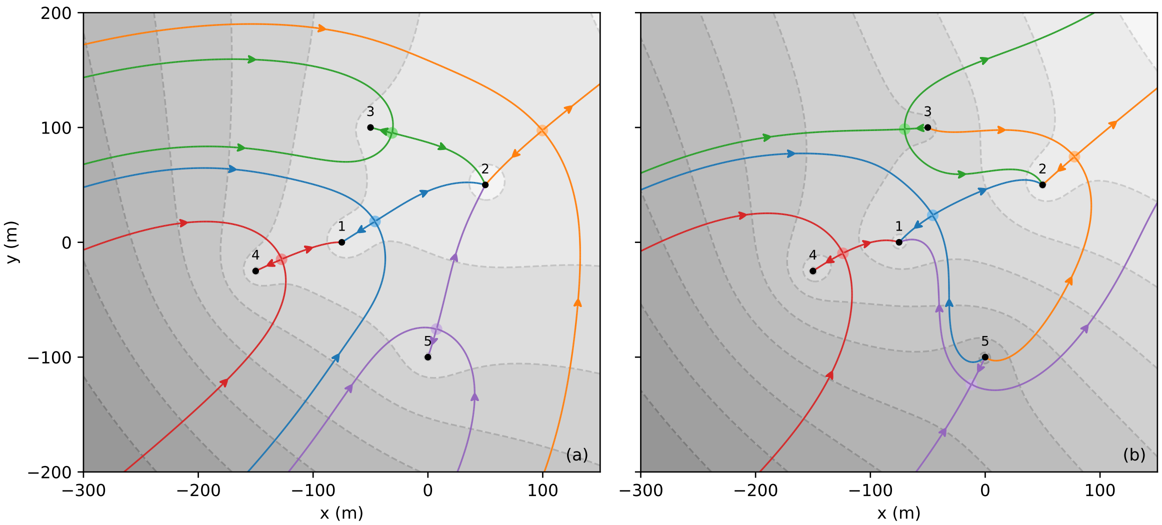

Figure 1: Example of five wells in uniform background flow. Equipotentials (filled grey contours, darker grey is higher potential), capture zone envelopes (colored), and stagnation points (light colored dots) for (a) Case 1 with five extraction wells, and (b) Case 2 with three extraction and two injection wells.

Uniform areal recharge may result in many different flow fields depending on the boundary conditions. Here, the boundary conditions are considered to be far away and result locally in straight head contours, circular head contours, or elliptical head contours. These three flow fields are referred to here as linear recharge, circular recharge, and elliptical recharge, respectively.

First, consider linear recharge that results in flow in the x direction only with a groundwater divide along the y-axis. The potential may be obtained from integration of (14) as

$$\Phi = -\frac{N_r}{2}x^2 + \Phi_0, \tag{15}$$

where Φ_0 is the potential along $x = 0$. The corresponding components of the discharge vector are

$$Q_x = N_r x, \qquad Q_y = 0. \tag{16}$$

For flow fields with areal recharge, the discharge potential fulfills Poisson's equation, which means that it may not be represented as the real part of a complex potential. The components of the discharge vector may, however, be combined into a complex discharge vector. For example, for linear recharge (16), the complex discharge vector may be written as

$$W = Q_x - \mathrm{i}Q_y = \frac{N_r}{2}(z + \bar{z}). \tag{17}$$

The discharge vector is now written as a function of z and its complex conjugate $\bar{z} = x - \mathrm{i}y$ (note that $(z + \bar{z})/2 = x$). The formulation of a vector field in terms of z and $\bar{z}$, rather than x and y, and the associated mathematical theory, is known as Writinger calculus. An introduction on the use of Wirtinger calculus in the field of groundwater flow is given by [22, 23]. The formulation of the discharge vector for a field of uniform areal recharge in terms of z and $\bar{z}$ facilitates the identification of stagnation points for multiple wells in a field of uniform areal recharge, as will be shown in the following. No advanced concepts of Wirtinger calculus are used in this paper.

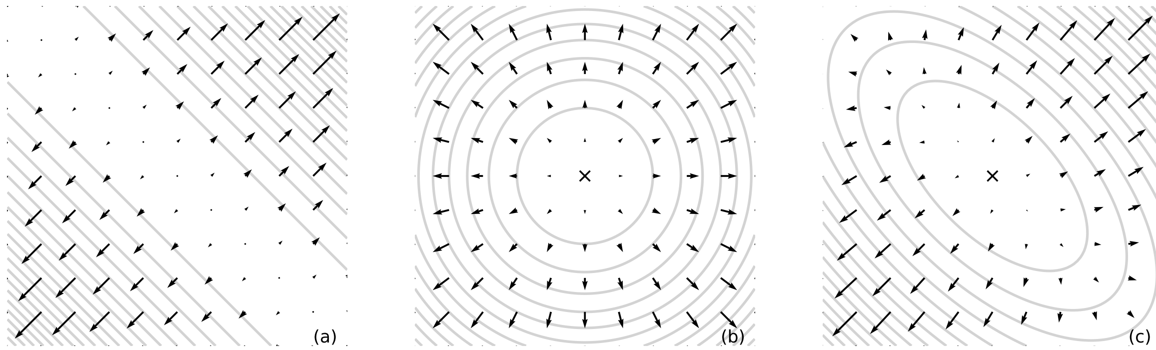

Figure 2: Examples of uniform areal recharge. Equipotentials (grey), discharge vectors (arrows), and high points (cross) for (a) straight equipotentials (linear recharge) for flow in direction $\alpha = \pi/4$, (b) circular equipotentials (circular recharge), and (c) elliptical equipotentials (elliptical recharge).

Equation (17) represents flow in the x-direction only. The discharge vector for linear recharge that results in flow in a direction that makes an angle α with the positive x-axis is

$$W = \frac{N_r}{2}\left(\mathrm{e}^{-2\alpha\mathrm{i}}z + \bar{z}\right). \tag{18}$$

For (18), the groundwater divide is a straight line that passes through the origin and makes an angle α with the positive y-axis. The flow field for the case that $\alpha = \pi/4$ is shown in Figure 2a. The location of the groundwater divide may be shifted by superimposing an appropriate solution for uniform flow ($W = W_u$).

The discharge potential corresponding to (18) may be obtained with Wirtinger calculus as (see Appendix B)

$$\Phi = -\frac{N_r}{8}(\mathrm{e}^{-2\alpha\mathrm{i}}z^2 + \mathrm{e}^{2\mathrm{i}\alpha}\bar{z}^2 + 2z\bar{z}) + \Phi_0. \tag{19}$$

Note that Φ is a real function of both z and $\bar{z}$. The expression for the discharge potential is not needed for the analysis in this paper and is only presented for completeness. The potential is contoured in the coming figures to aid in the understanding of the flow field.

Uniform areal recharge that results in circular or elliptical head contours may be obtained through superposition of two solutions (18) with different angles α_1 and α_2 and potentially different recharge rates N_1 and N_2

$$W = \frac{N_1}{2}\left(\mathrm{e}^{-2\alpha_1\mathrm{i}}z + \bar{z}\right) + \frac{N_2}{2}\left(\mathrm{e}^{-2\alpha_2\mathrm{i}}z + \bar{z}\right). \tag{20}$$

Note that the total areal recharge is $N_r = N_1 + N_2$. For example, the discharge vector for circular recharge is obtained by substituting $N_1 = N_2 = N_r/2$ and $\alpha_1 = 0$, $\alpha_2 = \pi/2$ into (20), which gives

$$W = \frac{N_r}{4}(z + \bar{z}) + \frac{N_r}{4}(-z + \bar{z}) = \frac{N_r}{2}\bar{z}. \tag{21}$$

The flow field has one stagnation point (Figure 2b), but it is not a saddle point. Groundwater flows away from this stagnation point in all directions and is referred to as a high point. The high point is located at $(x, y) = (0, 0)$ and indicated with a black cross. The high point may be shifted to an arbitrary location (x_0, y_0) by subtracting a uniform flow $W_u = N_r(x_0 - \mathrm{i}y_0)$ from the solution, which gives

$$W = \frac{N_r}{2}(\bar{z} - \bar{z}_0), \tag{22}$$

where $\bar{z}_0 = x_0 - \mathrm{i}y_0$.

As a second example, consider elliptical recharge at a rate N_r where the major principle axis of the ellipse makes an angle $\pi/4$ with the positive y-axis. The aspect ratio of the ellipse is governed by the ratio of N_1 and N_2 in (20). For example, consider the case that $N_1 = 4N_r/5$, $\alpha_1 = \pi/4$, and $N_2 = N_r/5$, $\alpha_2 = -\pi/4$. Substitution of these values in (20) and gathering terms gives

$$W = -\frac{3N_r}{10}\mathrm{i}z + \frac{5N_r}{10}\bar{z}. \tag{23}$$

The flow field has a high stagnation point at the origin and is shown in Figure 2c. More elongated ellipses can be obtained by increasing N_1 and decreasing N_2.

In summary, the general form of the discharge vector for uniform areal recharge (20) and a uniform flow term may be written as

$$W = az + b\bar{z} + c, \tag{24}$$

where a, b, and c are parameters defined as (note that a and c may be complex):

$$a = \frac{N_1}{2}\mathrm{e}^{-2\alpha_1\mathrm{i}} + \frac{N_2}{2}\mathrm{e}^{2\alpha_2\mathrm{i}}, \tag{25}$$

$$b = \frac{N_1 + N_1}{2}, \tag{26}$$

$$c = W_u. \tag{27}$$

5 Stagnation Points for Wells in a Field of Uniform Areal Recharge

The discharge vector for N wells in a background flow of uniform areal recharge is obtained by superimposing (24) and the discharge vector for N wells (the last term of (8))

$$W = az + b\bar{z} + c - \sum_{n=1}^{N} \frac{Q_n}{2\pi}\frac{1}{z - z_n}. \tag{28}$$

Setting $W = 0$, using (9) for the last term of (28), and multiplication with P gives

$$aPz + bP\bar{z} + cP - F = 0, \tag{29}$$

where P and F are given by (11) and (10), respectively. Recall that P and F are polynomials in z of order N and $N-1$, respectively, so that the first, third and fourth term of (29) are polynomials in z of order $N+1$, N, and $N-1$, respectively. Unfortunately, the second term is a polynomial in z of order N multiplied by $\bar{z}$, the complex conjugate of z, which complicates finding the roots of (29).

The roots (the stagnation points) of (29) may be found by first recognizing that when (29) equals zero, then so must its complex conjugate

$$\bar{a}\bar{P}\bar{z} + \bar{b}\bar{P}z + \bar{c}\bar{P} - \bar{F} = 0. \tag{30}$$

Elimination of $\bar{z}$ between the two polynomials (29) and (30) results in a polynomial equation in z of degree $(N+1)^2$. The roots of the original polynomial (29) are among the roots of this polynomial.

Solving (29) for $\bar{z}$ gives

$$\bar{z} = \frac{F - aPz - cP}{bP}. \tag{31}$$

$\bar{F}$ is obtained as the complex conjugate of (10)

$$\bar{F} = \sum_{n=1}^{N} \frac{Q_n}{2\pi} \prod_{\substack{m=1 \\ m \neq n}}^{N} (\bar{z} - \bar{z}_m). \tag{32}$$

Substitution of (31) for $\bar{z}$ in (32) gives

$$\bar{F} = \sum_{n=1}^{N} \frac{Q_n}{2\pi} \prod_{\substack{m=1 \\ m \neq n}}^{N} \frac{F - aPz - cP - \bar{z}_m bP}{bP}. \tag{33}$$

This equation is simplified to

$$\bar{F} = \frac{F^*}{b^{N-1}P^{N-1}}, \tag{34}$$

where

$$F^* = \sum_{n=1}^{N} \frac{Q_n}{2\pi} \prod_{\substack{m=1 \\ m \neq n}}^{N} (F - aPz - bP\bar{z}_m - cP). \tag{35}$$

Similarly, $\bar{P}$ is obtained as the complex conjugate of (11), which may be written as

$$\bar{P} = \frac{P^*}{b^N P^N}, \tag{36}$$

where

$$P^* = \prod_{n=1}^{N} (F - (az + b\bar{z}_n + c)P). \tag{37}$$

Substitution of (31), (34), and (36) for $\bar{z}$, $\bar{F}$, and $\bar{P}$, respectively, in (30) gives

$$\frac{\bar{a}(F - aPz - cP)}{bP} \frac{P^*}{b^N P^N} + \frac{\bar{b}P^* z}{b^N P^N} + \frac{\bar{c}P^*}{b^N P^N} - \frac{F^*}{b^{N-1}P^{N-1}} = 0. \tag{38}$$

Finally, multiplication with $b^{N+1}P^{N+1}$ and gathering terms results in a polynomial in z of order $(N+1)^2$

$$(\bar{a}F - \bar{a}aPz - \bar{a}cP + \bar{b}bPz + \bar{c}bP)P^* - b^2P^2F^* = 0. \tag{39}$$

The roots of this polynomial may be found again by using the polynomial package of numpy. The stagnation points are among these roots and are found by evaluating the discharge vector (28) at these roots and selecting the ones where the discharge vector equals zero. High points may be differentiated from saddle points by evaluating the discharge potential around the stagnation point (the potential at a high point is a local maximum). Only saddle points are used to delineate the capture zone envelope. The outlined procedure is again implemented in a short Python program and several examples are presented in the remainder of this paper.

6 Example: One Well in Three Types of Uniform Areal Recharge

As a first example, the capture zone envelope is delineated for a single well in the three types of uniform areal recharge shown in Figure 2. The results are shown in Figure 3; saddle points are indicated with a light colored dot and high points with a black cross. For all three examples, the area inside the capture zone envelope is the same: the total infiltration inside the capture zone envelope is equal to the discharge of the well. In Figure 3a, the background flow is linear recharge and there are two saddle points. In Figure 3b, the background flow is circular recharge and there is one saddle point and one high point. In Figure 3c, the background flow is elliptical recharge. For this case, there are two saddle points and two high points.

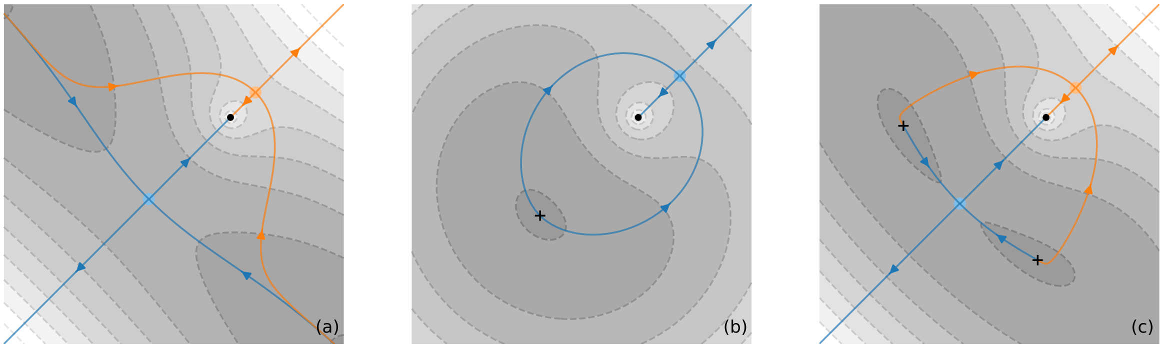

Figure 3: Examples of a single well in uniform areal recharge. The head contours for the background flow are shown in Figure 2. Equipotentials (filled grey contours, darker grey is higher potential), capture zone envelopes (colored), stagnation points (light colored dots), and high points (crosses) for a well in background flow that is (a) linear recharge for flow in direction $\alpha = \pi/4$, (b) circular recharge, and (c) elliptical recharge.

7 Example: Multiple Wells in Uniform Areal Recharge

For the last examples, consider the five pumping wells of Case 1 in Table 1. Capture zones are delineated for these five wells in two different types of uniform areal recharge. First, consider circular recharge at a rate $N_r = 2$ mm/d centered at the origin. The capture zone envelopes are shown in Figure 4. There are five saddle points and one high point.

Second, consider elliptical recharge with $N_1 = 0.5$ mm/d, $\alpha_1 = 0$, $N_2 = 1.5$ mm/d, and $\alpha_2 = \pi/2$. The elliptical recharge is centered at $(x, y) = (0, -200)$ by specifying $W_u = -200N_2 = -0.3\text{i}$ m^2/d. The capture zone envelopes for this case are shown in Figure 5. For this case there are also five saddle points and one high point.

The five combined capture zones cover an area equal to 250,000 m^2 for both cases. The total infiltration on this area is equal to the combined discharge of the five wells ($Q_{\text{tot}} = 500$ m^3/d). The capture zones are quite different for the two cases. For example, in Figure 4, the capture zone of well 1 consists of four sections, between the capture zones of the other four wells. In Figure 5, the capture zones of well 1 and well 3 both consist of three sections.

8 Conclusions

A new procedure was outlined to delineate the capture zone envelopes of multiple wells in a background flow of uniform areal recharge. Three types of uniform areal recharge were considered: fields that have straight head contours (referred to as linear recharge), circular head contours (circular recharge), and elliptical head contours (elliptical recharge). The streamlines that form a capture zone envelope pass through a saddle stagnation point. The stagnation points for multiple wells in an otherwise uniform flow field can be determined as the roots of a simple complex polynomial (13) with just a few lines of Python code (Figure A1). It was shown that for the case with uniform areal recharge, the stagnation points can be found among the roots of a more complicated complex polynomial (39), but the implementation in Python is still straightforward (all Python code is available on `github.com/mbakker7/capzones`). The capture zone envelope for an extraction well in uniform areal recharge has a finite area such that the total amount of recharge inside the capture zone envelope is equal to the discharge of the well. The presented approach can be applied to delineate detailed capture zone envelopes for complicated well fields in uniform areal recharge. The method of images may

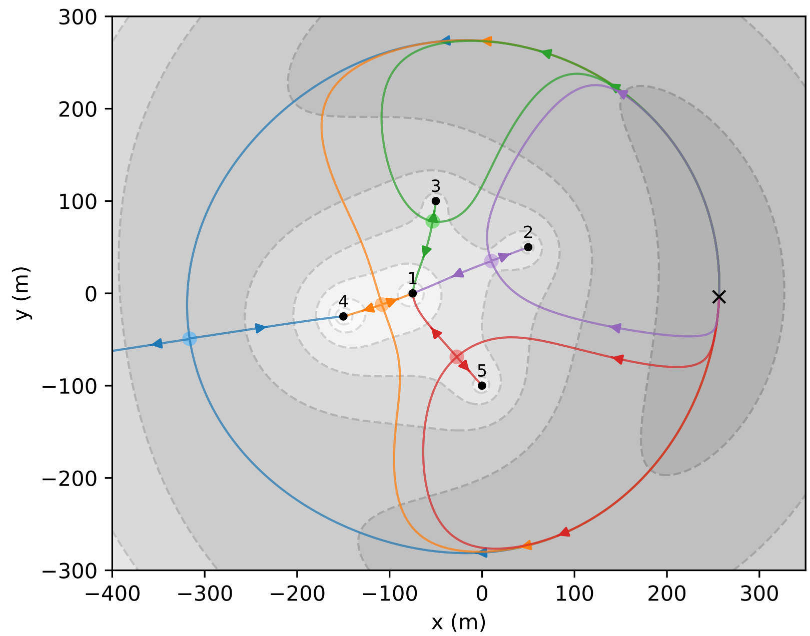

Figure 4: Examples of five wells (Case 1 in Table 1) in circular recharge background flow centered at the origin. Equipotentials (filled grey contours, darker grey is higher potential), capture zone envelopes (colored), five saddle points (light colored dots), and one high point (cross).

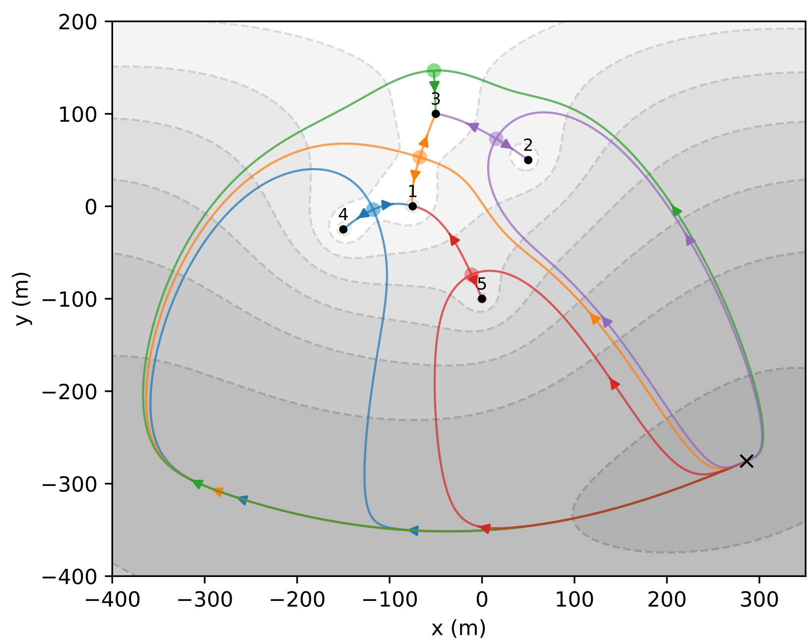

Figure 5: Examples of five wells (Case 1 in Table 1) in elliptical recharge background flow centered at $(x, y) = (0, -200)$. Equipotentials (filled grey contours, darker grey is higher potential), capture zone envelopes (colored), five saddle points (light colored dots), and one high point (cross).

be applied to include boundaries provided that the areal recharge also complies with these boundary conditions.

Acknowledgements

The approach to find the roots of polynomial (29) was inspired by an answer to a related problem of user dxiv on mathematics stackexchange.

Data Availability

All Python code for this paper is available at `github.com/mbakker7/capzones`.

Author Contributions

MB: Conceived and designed the analysis; Performed the analysis; Wrote the paper.

A Python Implementation

A Python function is presented to compute the stagnation points of an arbitrary number of wells in uniform background flow (Figure A1). The function builds the complex polynomial (13), and computes and returns the roots of the polynomial, which represent the complex locations of the stagnation points. In Equation (13), P is a polynomial defined by its roots (the locations z_n of the wells), and F is a polynomial that is the sum of polynomials defined by their roots. Polynomial (13) is obtained by multiplying polynomial P by W_u and subtracting polynomial F.

```
def stagnation_points(zw, Qw, Wu):
    '''
    zw : locations of wells, complex array
    Qw : discharges of wells, real array
    Wu : uniform background flow, complex
    '''
    poly = Wu * npp.Polynomial(npp.polyfromroots(zw))
    index = np.arange(len(zw))
    for n in range(len(zw)):
        poly -= Qw[n] / (2 * np.pi) * npp.Polynomial(
            npp.polyfromroots(zw[index[index != n]]))
    return poly.roots()
```

Figure A1: Python function to compute the stagnation points of an arbitrary number of wells in uniform background flow. The function returns an array of the complex locations of the stagnation points.

B Discharge Potential

The discharge potential (19) corresponding to the discharge vector (18) is derived. It is known from Wirtinger calculus that the discharge potential is related to the discharge vector as [e.g., 23]

$$W = -2\frac{\partial \Phi}{\partial z}. \tag{40}$$

Recall that z and $\bar{z}$ are independent variables in Wirtinger calculus. Integration of (18) gives

$$\Phi = -\int \tfrac{1}{2} W \mathrm{d}z = -\frac{N_r}{4}\left(\mathrm{e}^{-2\alpha \mathrm{i}}\frac{z^2}{2} + \bar{z}z + f(\bar{z})\right). \tag{41}$$

The function $f(\bar{z})$ is chosen such that $\Phi = 0$ along the water divide, i.e., along the line $y = -x/\tan(\alpha)$, which may be written in terms of z and $\bar{z}$ as

$$z = -\mathrm{e}^{2\alpha \mathrm{i}}\bar{z}, \tag{42}$$

where it is used that $x = (z+\bar{z})/2$ and $y = (z-\bar{z})/(2\mathrm{i})$ Substitution of (42) for z in (41) and setting the result to zero gives

$$f(\bar{z}) = \tfrac{1}{2}\mathrm{e}^{2\alpha \mathrm{i}}\bar{z}^2. \tag{43}$$

Finally, substitution of (43) for $f(\bar{z})$ in (41) gives (19).

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