Mathematics and Sports

A mathematical model for scoring athletic performances

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Abstract
We present an application of our recently proposed scoring method to running performances. To this end we use the performances corresponding to the elite level, as given by the World Athletics scoring tables, in order to calibrate the high-end (max-score) of our scoring formula. For the lower end (null-score) we start from the prescription put forward by one of the authors (G.P.), stated epigrammatically as "walking is not running" and obtain an estimate of the corresponding performances. The result is a set of parameters, calculated once and for all, which allow, given the distance and the registered time, to obtain the scoring for the performance for distances ranging from sprints to ultramarathons (for both sexes and adjusted for the age of the runner).

Keywords: scoring, running performances, athletics, age factors

1 Introduction

Humans are naturally attracted to competition. The domain of sports is special in the sense that the contest is not motivated by a scarcity of resources but rather pursues the attaining of a goal (usually in the form of a personal best, and, for the elite, a record). Competition is a central dynamic in most (if not all) sports. According to R. Mertes [Martens 1976] competition is a form of social evaluation. Viewed under this angle, the term competition encompasses not only the comparison of individuals against one another but also against some objective standard of excellence. Organized competition brings order to the process since it fixes the rules defining the means which are allowed in order to achieve a goal.

Looking back at the origins of organized sport in Ancient Greece [Golden 2004] one encounters the first facet of competition, that of the comparison to others. In fact, in the Ancient Greek tradition [Miller 2004] competitions were governed by the quest for excellence, where the most important thing was victory [Leonard 2016]. This had as consequence that no precise measurements were required, and neither the distances nor the implement weights were standardized. In fact, very few data on the performances of the ancient Greek athletes have been preserved. However, over the centuries, the situation evolved and the technological progress helped make sports more quantitative. Athletics profited from the increasing accuracy in time and distance measurements.

Having precise measurements facilitates also the second facet of competition, where the aim is not the comparison to others but rather the accomplishment of a self-fixed goal. Thanks to the measurements one has a way to assess one's level of ability and appreciate the progress towards the fixed goal. Athletics is *par excellence* a quantitative sport where the performance is expressed in time or length allowing thus straightforward comparisons within a specific event. But once one wishes to compare performances obtained in two different events, the mere result of a precise measurement is not sufficient. This is where scoring tables can be valuable. They made their appearance in Athletics at the end of the 19th century in order to allow the classification in combined events [Zarnowski 1989]. However, the question of performance scoring can be asked in a more general setting. How does one attribute a score for a performance in a sport where precise measurements are not really helpful (like mountain climbing or open-sea sailing)? D. Harder addressed this question [Harder 2001] and his answer, to put it in a nutshell, is that "two performances are equivalent when the fraction of the practitioners who obtain them is comparable". The mathematical foundation for the construction of scoring tables has been set by one of the co-authors in [Grammaticos 2007] and served as a basis for the recent proposal [Grammaticos-Meloun-Purdy 2022] of a scoring system extending the one proposed by another co-author [Purdy 1974-77]. The idea behind our approach is that the score of a performance should be linked to the inverse of the probability of exceeding this performance. The best, practical, way to assess this probability is through the statistics of the distribution of performances establishing, thus, a link between our scoring approach and that of Harder.

As already explained (and, in fact, announced in the title of the paper) this work will focus on scoring performances in athletics, and more specifically on running. Scoring the athletic performance is far from trivial. Several questions must be answered before setting up a scoring system and, what is worse, the various choices exert influence upon each other. In order to construct a well-balanced and fair scoring one must first determine what is the performance warranting an arbitrarily fixed high mark (typically we are talking here about 1000 points), the max-score, and what is the performance below which one will obtain zero points, the null-score. Once this is done for one event it has to be done for every other one taking particular care that the performances corresponding to 1000 and 0 points are indeed equivalent. Even when this is ensured, there is no guarantee that the intermediate performances will be equivalent all along the scoring table. This depends on how the performances are distributed for the various events and the simplifying, tacit, assumption of scoring tables specialists is that these distributions follow indeed the same law for all events.

An essential requirement for fair scoring is that it be progressive [Trkal 2003]. In practice ‘progressive’ means that the same increase in performance garners a larger increase in points when the performance increases. For instance, a gain of 1 second in 400 m will bring more points when it corresponds to going from 46 s to 45 s compared to the case when the athlete goes from 58 s to 57 s. The progressive character of scoring is a *sine qua non* for fair scoring. It was already a main feature of the tables proposed by one of us [Purdy 1972] and will be present in the tables that will be presented in the next section.

One caveat is necessary at this point. Everybody has their preferred events. Nobody will score equally well in short-, middle- and long-distance events. Having a good mark in some range of distances and so-so marks outside this range is natural and in no way an indication of unfair scoring. To use a striking example, a sprinter may be excellent in short-distance running but will score poorly over longer distances and even may well be unable to finish a marathon. Thus, it’s important to realise that the scoring function represents a statistical summary of the entire population and not a representation of any one member of the population. With that said, the scoring function is useful by individual members of the population for measuring the value of their performance.

An important remark concerns the differences between men and women. When it comes to running events women are roughly 10 % slower than men. And this difference is more or less constant over the whole spectrum of running events, with distances covering several orders of magnitude. A fair scoring must account for this difference and attribute the adequate number of points.

Finally a fair scoring method should take into account the variations of the performance with age. In order to account for this one must introduce appropriate age factors which correct the performance converting it, in some sense, to the performance the athlete would have registered in his prime.

The present paper will focus on scoring running performances covering the whole spectrum, from very short sprints to multi-day ultramarathons. We shall start with a short presentation of our scoring system explaining how one can use data in order to fix the parameters of the scoring formula. We shall then explain how we have extracted those data from the World Athletics tables [Spiriev 2022] and the best performances on ultra-running [Wikipedia] allowing to fix the upper part of the tables. The approach presented in the work of one of us, G.P., is used as a basis for the determination of the null-score, the latter being another necessary ingredient for the determination of the parameters. The result is a closed-form mathematical formula which allows to compute the scoring for *any* distance between 50 m and over 1000 km from the simple datum of the time to cover the distance (as well as the sex and the age of the runner). One of the authors, G.P., developed the brand name “TraxScore” a number of years ago as a vehicle to help road race organizers, coaches and runners have a handle by which they could represent their performance from any distance and time. It will be used throughout this paper.

2 The new scoring system

In a recent publication [Grammaticos-Meloun-Purdy 2022] we proposed a new scoring system, which is an evolution of the one presented in the work of one of the authors [Purdy 1974-77]. We are not going to present the details of the construction of the new system but limit ourselves to a succinct presentation with emphasis on the determination of the parameters.

The scoring tables introduced in [Grammaticos-Meloun-Purdy 2022], are given in terms of a simple mathematical formula involving the number of points p and the performance u which, in our case, is the normalized mean velocity obtained during a running event

$$p = a(e^{bu} - 1) + f \log\left(1 + c(e^{du} - 1)\right) \tag{1}$$

where a, b, c, d, f are parameters. The first term of (1), as explained in [Grammaticos-Meloun-Purdy 2022], corresponds to the Gompertz part of the distribution of performances, with b controlling the rate of growth of scoring with the velocity. The factor a adjusts the contribution of this term to the total score (up to a global scaling). The second term, corresponds to the skewed logistic part of the performance distribution. Again, the parameter d controls the rate of growth of scoring with the velocity (and its effect is felt mainly for low velocities). Quite expectedly, f calibrates the contribution of this term to the score.

Obviously, we could have constructed the scoring formula in terms of the time t registered during the running event over a distance s, but we have opted for the velocity since the latter is the natural physical quantity through which to assess the athlete’s effort. The velocity has been normalized so as to make the parameters b and d dimensionless. Starting from the mean velocity $v = s/t$, obtained during the running event, we divide it by the value v_m corresponding to the value that obtains the maximum number of points. Thus the normalized velocity $u = v/v_m$ can be understood as $u = t_m/t$ where t_m is the time that obtains the maximum number of points. Speaking about the latter, it is clear that an overall scaling of the scoring points can be applied depending on whether we wish to score over 1000 points or any other value. For instance, if we have the coefficients fixed so as to attribute 1000 points to a performance $u = 1$ and decide to attribute 1400 points to this performance, it suffices to multiply a and f by a factor 1.4. (It goes without saying that it is possible to apply the scoring formula to values of u exceeding 1, i.e. corresponding to velocities higher than v_m).

In Figure 1 we present a graphic of the scoring curve where both the scoring points and the performance (normalized mean velocity) have their maximum fixed to 1.

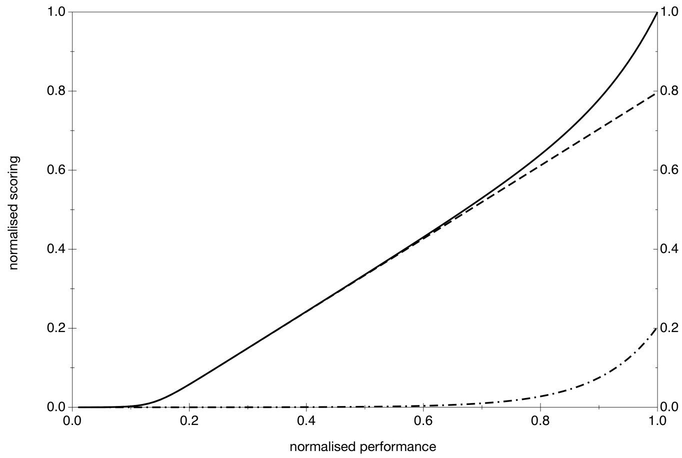

Figure 1. *An example of normalized scoring. The solid line represent the total number of points, the dashed line corresponds to the part involving the logarithm and the dot-dashed line comes form the exponential part*

Using this curve as a guide we can now explain the procedure for the fitting of the parameters. We see that the exponential part is negligible for the very low performances. This is a choice of ours, so as to have the fast increasing part entering only at higher performances thus allowing us to fix separately the parameters of the two parts of the scoring equation.

In [Grammaticos-Meloun-Purdy 2022] it was decided that the parameter c can be fixed once and for all to a value in the 0.01-0.0001 range (corresponding roughly to the fraction of the population unable to register any performance). For simplicity, we are going to work here with a value $c = 0.001$. Second, we remark that the curve obtained from (1) starts from the origin of the coordinates, i.e. zero points correspond to zero performance. Thus the null-score normalized performance $z = v_0/v_m$, being finite, will inevitably score a non-zero number of points q. However we can make this number of points as small as we like, for instance something between 1 and 10 when the maximum score is 1000 (or between 0.001 and 0.01 if we normalize the maximum score). This leads to a first relation

$$f \log\left(1 + c(e^{dz} - 1)\right) = q. \tag{2}$$

The effect of the term involving the logarithm on higher performances can be seen in Figure 1: the number of points grows practically linearly with the performance.

Next we turn to the exponential term. As explained in [Grammaticos-Meloun-Purdy 2022], the parameters a and b can be fixed in a very simple way if one decides what is the contribution of this term at a performance half of the maximum one, i.e. $u = 1/2$ and at the maximum $u = 1$. We can assume that at $u = 1/2$ we have a very small number r and that at the maximum $u = 1$ the contribution of the exponential term is w. (The choice of the value of w is arbitrary. In fact a value of w in the 0.20-0.25 range, as in Figure 1, is guaranteed to provide a, perfectly acceptable, moderately progressive scoring). The parameters a and b are then given by the expressions $a = r^2/(w - 2r)$ and $b = 2\log((w - r)/r)$. Finally we use the fact that the maximum number of points m (in principle normalized to 1), obtained for $u = 1$ is the sum of the contributions of the exponential term, which as we saw is equal to w, and the logarithmic term. This allows us to write a second equation involving f and d:

$$w + f \log\left(1 + c(e^{d} - 1)\right) = m. \tag{3}$$

Solving (2) and (3) allows us to obtain f and d. However given the form of the equations the solution can only be obtained numerically (or graphically). In fact, eliminating between (2) and (3) we obtain an equation for d

$$\log\left(1 + c(e^{d} - 1)\right) = \left(\frac{m - w}{q}\right) \log\left(1 + c(e^{dz} - 1)\right), \tag{4}$$

represented graphically in the Figure 2,

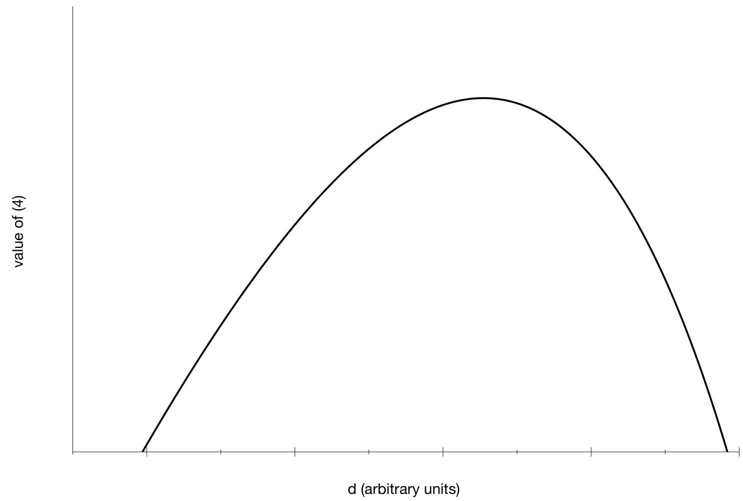

Figure 2. *Graphical solution of equation (4).*

Note that the root that is of interest to us is the largest of the two.

In practice we must decide on the value of the contribution w of the exponential term and the two very small values q and r. The null-score performance z will be determined starting from a prescription given by one of us in [Purdy 1974-77]. Moreover we shall need the value of the velocity v_m that corresponds to the maximum score, something that has to be done based on a compilation of actual results and a statistical treatment thereof. We are going to address the two last points in the next section.

3 Determining the max-score and the null-score performances

Analyzing data from a large number of races, while undoubtedly worthwhile, is a resource-intensive task. It can and must be done when the goal is to produce a scoring formula aiming at replacing the existing ones. However, before embarking upon such an ambitious enterprise, it is essential that the whole scoring construction, delineated above, be validated. In order to do this one can rely upon trustworthy results already compiled and codified. After all, what we need at this point is just the velocity v_m corresponding to the maximum number of points, as a function of the race distance s (for both men and women). In fact, from a practical viewpoint, we expect the scoring formula we propose to be useful (and used) for a large population of recreational and competition runners.

The solution we have opted for, in order to adjust the parameters of our scoring formula, is to use the performances corresponding to a high number of points in the World Athletics tables [Spiriev 2022] (the ones compiled by B. Spiriev). We have thus obtained from the World Athletics tables the velocities, v_m, for distances ranging from 50 m to 100 km, corresponding to a score of 1200 points. The choice of 1200 points has been dictated by the fact that we wish to complement the list with ultra-marathon performances. For these very long distances, we decided to work with the world records (since they are the only easily available data) and attribute them 1200 points, which is slightly below the number of points of the world records for the "standard" track distances (a reasonable assumption given that the very long distances are relatively less popular than the distances up to the marathon).

In the three tables that follow we summarise the results obtained from the World Athletics tables and the ultramarathon world records. The first table contains the results for track events.

distance s (m)	MEN velocity v_m (m/s)	WOMEN velocity v_m (m/s)
50	8.83	8.24
55	9.02	8.38
60	9.20	8.50
100	9.98	9.09
200	9.94	8.95
300	9.52	8.43
400	8.95	7.98
500	8.52	7.63
600	8.14	7.17
800	7.68	6.78
1000	7.46	6.55
1500	7.03	6.23
2000	6.82	6.08
3000	6.58	5.85
5000	6.39	5.67
10000	6.11	5.40

The second table, also obtained from the World Athletics tables, gives the velocities for road events. Note that for the distances of 5000 and 10000 m there exist separate scorings for track and road events. However, given the increasing popularity of road events and the high level of performances registered there, the difference in scoring (in the most recent, 2022, tables) between track and road events is negligible.

distance s (m)	MEN velocity v_m (m/s)	WOMEN velocity v_m (m/s)
15000	5.97	5.36
20000	5.92	5.29
21097	5.89	5.28
25000	5.80	5.18
30000	5.69	5.08
42195	5.51	4.90
100000	4.44	4.14

The third table gives the velocities as a function of distance for ultramarathon events. Since some events are based on time duration rather than distance, we have separated accordingly the performances of men and women.

distance s (m)	MEN velocity v_m (m/s)	WOMEN velocity v_m (m/s)
50000	5.14	4.63
80450	4.62	3.94
85492 (6 hr)	-	3.96
97200 (6 hr)	4.50	-
100000	4.51	4.24
149130 (12 hr)	-	3.45
160900	4.12	3.52
177410 (12 hr)	4.11	-
270116 (24 hr)	-	3.13
309400 (24 hr)	3.58	-
397103 (48 hr)	-	2.32
473496 (48 hr)	2.74	-
883631 (6 days)	-	1.71
1000000	2.04	1.51
1036800 (6 days)	2.00	-
1609000	1.78	1.48

A first remark concerns the ratio of women to men mean velocities. This is a question that was addressed in a previous publication of one of the authors [Grammaticos-Charon 2014]. The conclusion there was that for distances up to the marathon, the ratio of women to men velocities is roughly 0.9. Perusing the results of the three tables we see that this ratio holds for distances up to 100 km. It diminishes beyond this point but we believe that this is not a physiological effect but rather one due to the fact that there are fewer women participating in ultramarathon events. In order to simplify our approach we shall assume that the ratio 0.9 holds over all distances and thus posit that the max-score velocity for women is 9/10 that of men for the same distance.

Having settled the question of women versus men we return now to the initial question, namely of how to obtain the variation of velocity as a function of the distance. Clearly this is a delicate matter given the fact that we pretend to cover a range where the ratio of distances (or of durations) of the longest to the shortest events is of the order of 10^4-10^5. It is clear that the physiological mechanisms entering the athlete's effort vary substantially along the whole spectrum of events and the function $v = f(s)$ can be quite complicated. In [Purdy 1974] one of the authors had presented a graphical representation of that function together with a best fit, albeit on distances not exceeding 100 km. In what follows we are going to distinguish five domains over which we shall obtain a relation between v and s. The first domain covers the distances from 50 to 300 m. This domain is governed by the fact that the athlete must accelerate from velocity zero up to his maximal velocity. This explains the left branch of the curve in Figure 3.

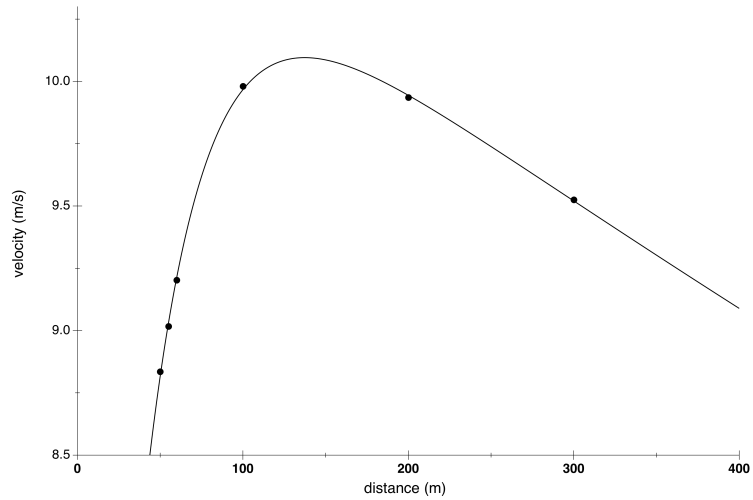

Figure 3. *Velocity as a function of distance for races spanning the interval 50-300 m.*

However the maximal velocity, once reached, cannot be maintained for long, the athlete depleting fast his alactic anaerobic energy reserves. The lactic anaerobic mechanism enters into play and is crucial in races spanning the distances from 300 to 1000 m, Figure 4.

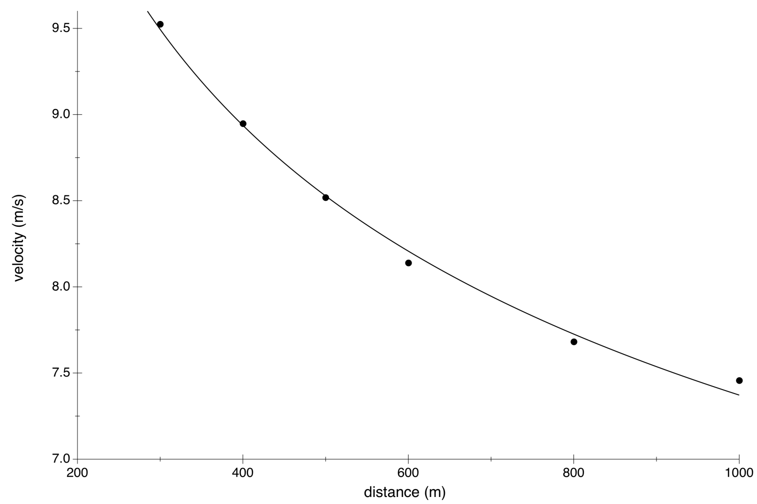

Figure 4. *Velocity as a function of distance for races spanning the interval 300-1000 m.*

Beyond these distances, and roughly up to the marathon, the main energy production mechanism is the aerobic one leading to the curve represented in Figure 5.

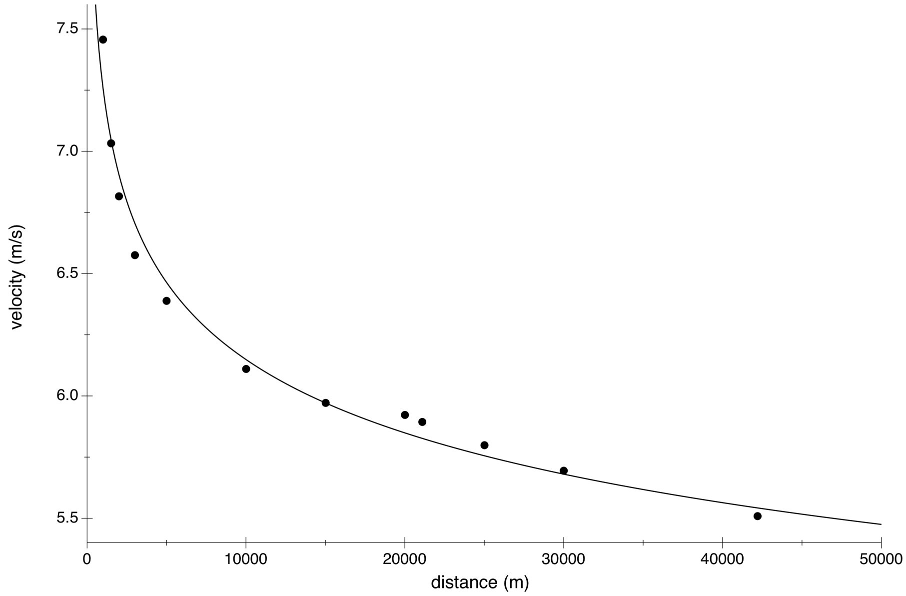

Figure 5. *Velocity as a function of distance for races spanning the interval 1000 m to the marathon.*

Once the duration of the effort exceeds two to two and a half hours the glycogen reserves of the organism are depleted and the aerobic mechanism relies now on the use of lipids. This results into a sharp decline of the velocity, as can be seen in Figure 6, covering the distances up to 24 hours.

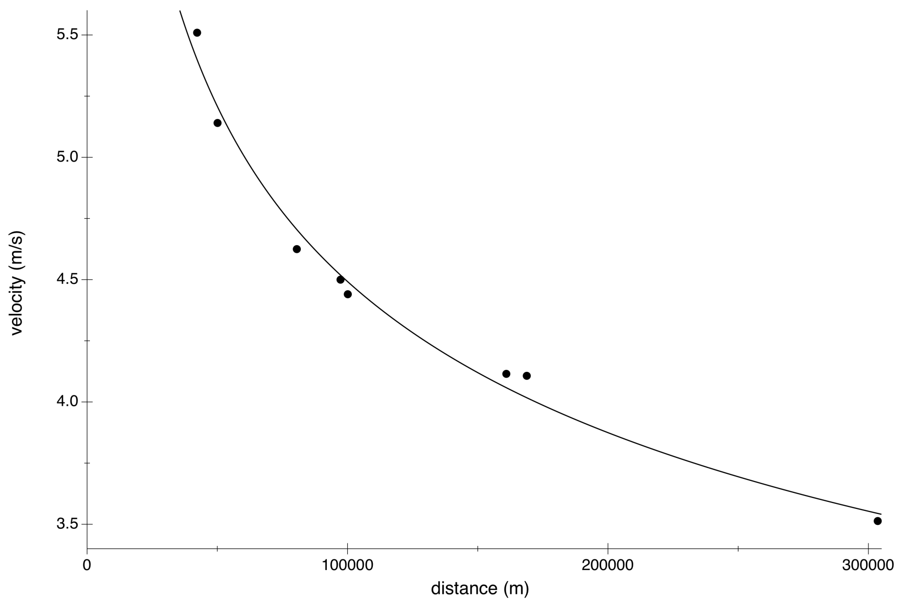

Figure 6. *Velocity as a function of distance for races from the marathon to 300 km.*

Finally for events that last for more than 24 hours, the athlete's effort is interspersed by unavoidable pauses corresponding to sleep, ingesting food and taking care of other bodily functions, resulting in a substantial decrease of the velocity, Figure 7.

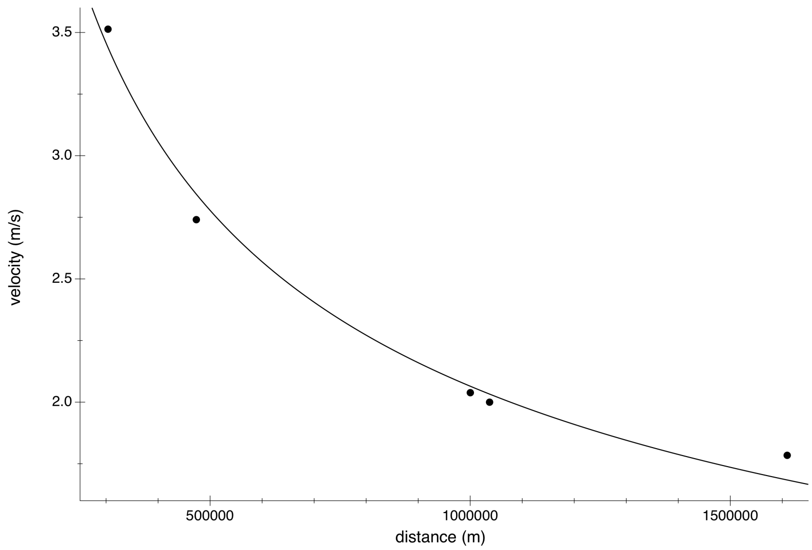

Figure 7. *Velocity as a function of distance for ultramarathons, exceeding 300 km.*

Having presented graphically the dependence of the velocity on the distance we turn now to fitting it with a simple mathematical formula so as to have an explicit expression allowing us to interpolate the results to any distance. The domain 50-300 m needs a special treatment due to the presence of the acceleration phase. It turned out that the data in Figure 3 can be represented nicely by the expression $v = C\exp(-\lambda s) - D\exp(-\mu s)$ where $C = 10.950$ m/s, $D = 7.298$ m/s, $\lambda = 4.658\ 10^{-4}\ \mathrm{m}^{-1}$ and $\mu = 0.0271\ \mathrm{m}^{-1}$, corresponding to the continuous line in the figure. For all distances exceeding 300 m, given the shape of the velocity curve we have opted for the simple expression $v = A/s^{\gamma}$. The fit of the data over each of the four domain, corresponding to Figures 4 to 7 yielded the following values: $A = 31.475$, 11.950, 52.425, 765.5 (in units of $\mathrm{m}^{1+\gamma}$/s) and $\gamma = 0.2101$, 0.0721, 0.2134, 0.4281 for the four domains respectively. Since for each domain the fitting curve is different it is important to obtain the anchoring points (i.e. the ones where the two curves cross each other) so as to have a continuous curve representing the variation of the velocity with the distance. The four anchoring points we obtained are 290.2 m, 1122.2 m, 35032 m and 265300 m. The result can be contained in a single figure (Figure 8) provided we use a logarithmic scale for the abscissa.

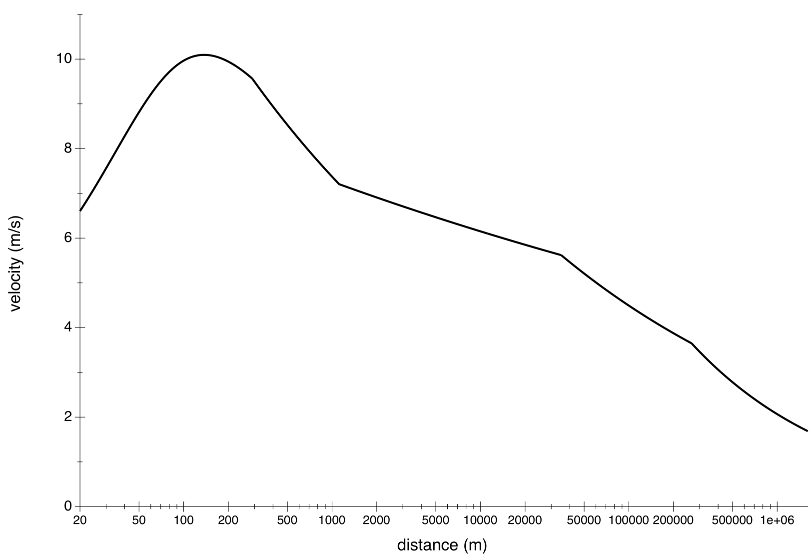

Figure 8. *Velocity as a function of distance for races from 20 m to 1000 miles.*

The changes in the slope of the curve allow one to visualise the changes in the energy production regimes. The discontinuities in the derivative of the curve are an artifact of our fit-per-domain, but they are inconsequential since we are interested only in the value of the function which varies continuously. Thus for any race distance from, say, 20 m to 1000 miles one can obtain through an elementary calculation the velocity corresponding to the max-score of the tables.

Having settled the question of the max-score we turn now to the determination of the null-score. One of the authors had addressed the question of the null-score in [Purdy 1974-77]. In the case of track or road events the argument is that walking is not running and an empirical formula was proposed, relating the null-score velocity to distance, $v_0 = 2 - s/10^5$. Unfortunately using this expression as such would have made impossible the simplifications we are aiming at, for the construction of our scoring formula. Still we are going to use this prescription as a basis for the determination of the null-score velocity. To this end we start from the null-score velocities given by the expression $v_0 = 2 - s/10^5$ for races from 5000 to 40000 m and we fit the points obtained with an expression $v_0 = B/s^\gamma$ where $\gamma = 0.0721$, i.e. the very same value of the exponent obtained from the fit of the max-score velocities in the range 1000 m to the marathon.

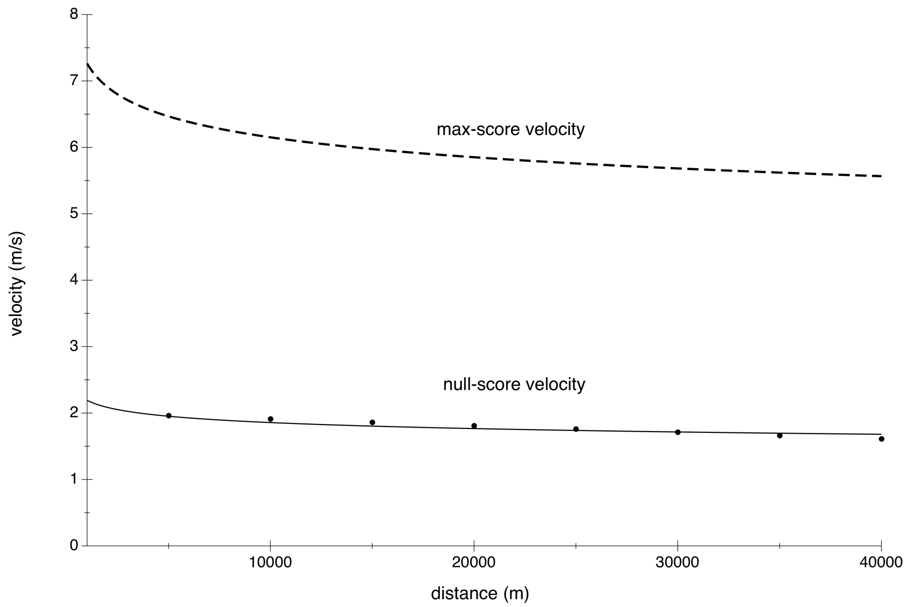

Figure 9. *Max and null-score velocities for races from 5 m to 40 km.*

The value of B turns out to be equal to 3.607 (in the appropriate units). It is thus straightforward to obtain the value of the normalized null-score performance by simply dividing B by the value of A for the same range of distances, resulting to $z = 0.302$. Having obtained this very simple result, we decided that the normalized null-score velocity will be equal to 0.3 for all distances. This choice, which may look somewhat arbitrary is of capital importance in the construction of the scoring formula. Having a value of z independent of the distance means that we can compute the parameters just once. The scoring formula is thus universal, having the same parameters for any distance.

A few remarks are in order at this point concerning the value of z. First, the "walking is not running" prescription would give a much smaller ratio, starting roughly at 0.2 for sprint events, increasing to slightly over 0.3 for longer distances and going progressively to 0 at 200 km. While we believe that the principle proposed in [Purdy 1974-77] is most realistic, its implementation in a wider range of running distances would necessitate further adjustments and, given the advantages of the fixed z value, it is simpler to renounce on its strict application. While the value of z is larger than the ones recommended in [Purdy 1974-77], it is still substantially smaller than the ones of the World Athletics tables, where they vary from 0.6 for sprint events to 0.4 for the 100 km race. We are convinced that using a smaller value of null-score will help a larger population of (amateur) athletes profit from obtaining a score for their performances.

4 The final scoring formula

The advantage of the assumptions made in the previous section concerning the max- and null-score velocities is that we can obtain a scoring formula that is universal. This means that its parameters can be defined once and be valid for any race distance s. Thus one can build an application where the user will input only the distance and the time (and specify the sex) and the result will be obtained by a simple application of the formula without having to recompute the scoring parameters.

On a practical level we decided to provide a scoring over 1000 points. It goes without saying that for performances better than the max-score one the number of points will exceed 1000. As explained in section 2, we can work with the nomalised value $m = 1$ and in the and multiply everything by 1000. In section 2 we proposed a normalized value for w in the range of 0.2-0.25. In what follows we fix its value to 0.25. As we saw in section 3 the null-score z is fixed to 0.3. A choice must be made for the two small values q and r. We choose for both of them the value 0.01. Using these values (and the fact thta c was fixed once and for all at 0.001) we can solve equation (4) obtaining the value $d = 16.358$. This gives for f the value $f = 0.07936$. Using their explicit expressions we obtain $a = 4.3478\ 10^{-4}$ and $b = 6.3561$.

In order to obtain the score for a given performance we start, using the distance s, by determining the domain in which to seek the max-score velocity v_m. From the performance expressed in time, t, we compute the velocity $v = s/t$ and obtain the normalized velocity $u = v/v_m$. If the athlete is a woman we use u divided by 0.9. It suffices then to use the value thus obtained in order to obtain from (1) the number of corresponding points. A word of caution is necessary at this point concerning very short distances. Although the scoring formula is valid from a purely mathematical point of view, we believe that the domain of sprints would necessitate a special treatment (and we shall deal with this in some future work of ours), while a blind application of some mathematical recipe may lead to unwarranted conclusions. Thus, to be on the safe side, we recommend some caution in the application of our scoring approach to distances of 50-150 m.

The procedure we just described is elementary and its practical implementation would necessitate just a few lines of code.

5 Introducing age factors

The athletic performance does vary with age, improving constantly through puberty and reaching a maximum around the early to mid-twenties. The level of performance can be maintained for some years but the decline is unavoidable, linked to the overall ageing of the organism, starting around the early to mid- thirties. A scoring approach designed for an as large as possible population, and not just the elite, should take into account the age variations of the performance level and reward the latter with the appropriate score.

The World Masters Athletics association has addressed the question of providing a fair comparison of the various age groups and proposes a solution in the form of age grading tables. Every performance is modified by the appropriate age-dependent factor allowing either inter-age comparisons or providing an adjusted performance to be scored by an a priori age-independent scoring method.

The study of age factors has been the object of a publication of one of the authors [Grammaticos 2009] where a simple mathematical expression was proposed. The most interesting result of that study was the fact that the evolution of performances follows a 'universal' curve (Figure 10) when expressed in terms the velocity for track events. (A similar result holds for field events when one considers the length of the jump or of the throw. In fact as shown in [Grammaticos 2020] the same behaviour is valid also for swimming provided one plots the energetic cost as a function of age).

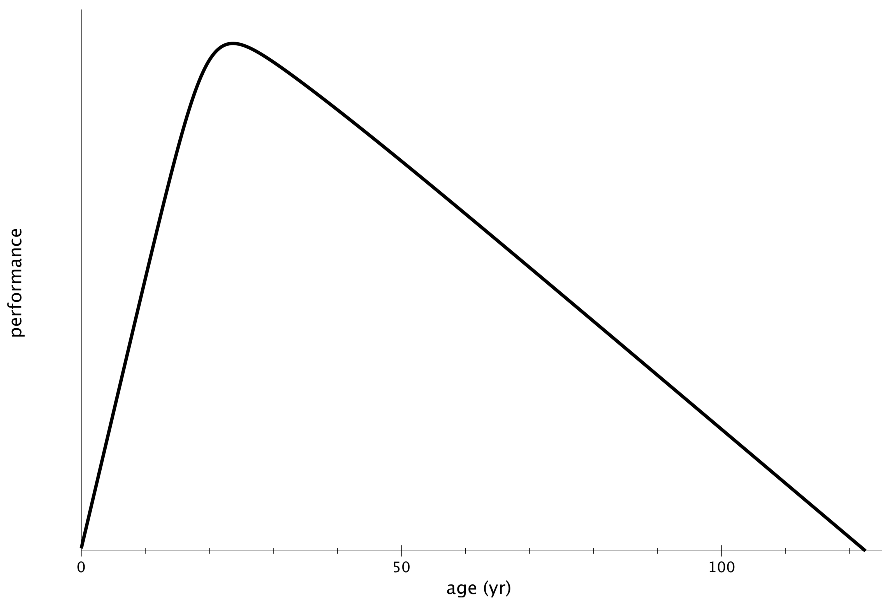

Figure 10. *The universal evolution of peformance with age.*

The performances are improving at a roughly constant page during childhood and the teens and start declining at a roughly five times slower pace when age advances. Using this result it is easy to propose age factors for the younger and for the older athletes.

In this paper we follow the age classification of World Athleitcs as far as young athletes are concerned and that of the World Masters Athletics for the older ones. In practice we shall apply as age factor to athletes below 23 years of age and to those of age 35 and above. The corresponding factors are shown in Figure 11.

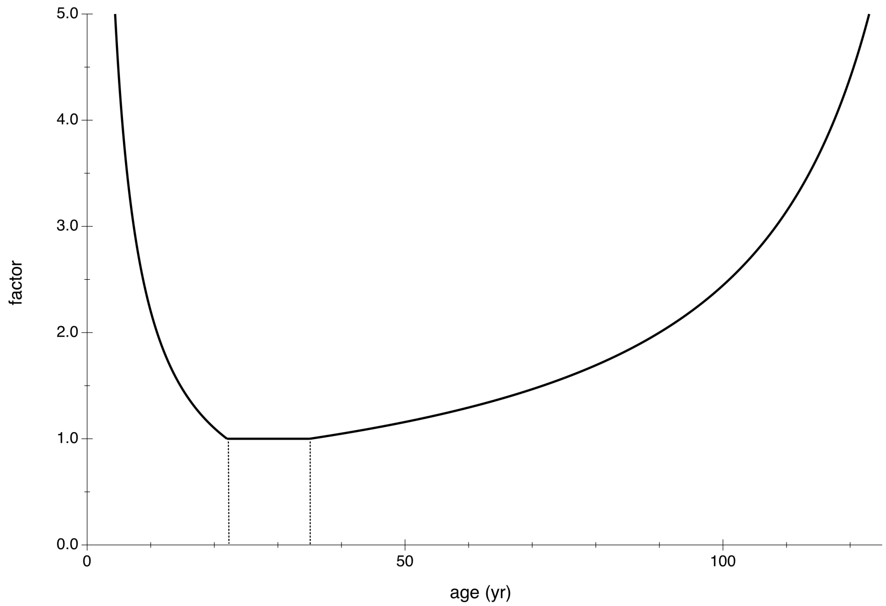

Figure 11. *Age factors used in this paper.*

A very simple parametrization was adopted here. For younger athletes, the expression we use in $F = 22/A$ where A is the age of the athlete in years. For the older ones, the corresponding expression is $F = 110/(145 - A)$. Once the age factor is computed it suffices to multiply it with the mean velocity of the race in order to obtain the adjusted performance to be used in the scoring formula.

6 Conclusion

The main objective of this paper was to show that the scoring tables we proposed in [Grammaticos-Meloun-Purdy 2022] can be applied to a real-life situation. To this end, we decided to provide scoring for running events, since running is an ubiquitous athletic activity, practiced by a substantial fraction of the active population and has an extreme variety as far as the distances involved are concerned. Our aim was not to introduce scoring tables for running aiming at replacing existing ones but rather to provide a proof-of-concept constructing a scoring formula starting from reliable, already statistically treated, data. To this end we chose the World Athletics tables (the ones covering all athletics events and not just the combined events ones) and decided to obtain the max-score performance of our scoring from the one corresponding to 1200 points in the World Athletics tables. The slight inconvenience of this approach is that no scoring of performances is available beyond the 100 km race. In order to palliate this we complemented the max-score performances obtained from the tables with the best performances for ultramarathon events.

Having the max-score velocity as a function of the distance we proceeded to fit it with a simple mathematical expression that accurately models human performance in running. We did not attempt an overall fit with a unique formula since we believe that fitting per range of distances is better adapted to the various physiological regimes which play the major role in the races in question. The null-score performance was obtained based on estimates by one of us (G.P.) leading in the end to a constant ratio of the null-score velocity to that of the max-score. In order to provide scoring tailored to the performance differences between men and women we decided to fix the max-score velocity for women to 90 % of that of men.

While the mathematical expression of our scoring formula may look a little bit complicated it has the advantage of being straightforward to implement. In fact the scoring is provided by the sum of two terms only one of which plays a role in low-to-moderate performances and which becomes very simple when the performances grow. The second term plays a role only at the high end of the spectrum and thus the two terms can be adjusted almost independently. This, combined with the fact that we have fixed the ratio of null-to max-score, allows to calculate the parameters of the scoring once and for all, making them in some sense universal. We have thus a scoring formula which is valid for *any* distance, say from 20 m to 2000 km, and which can be practically implemented with minimal effort.

This work is the fruit of a collaboration which aimed, first, at correcting mistakes in performance scoring and, second, at providing a scientific basis thereof, constituting thus a significant contribution to the science of modelling human performance. The initial motivation for such an enterprise came from the domain of combined events. Several mathematical models [Grammaticos-Meloun-Purdy 2022] have been proposed for their scoring over the years, none being devoid of drawbacks. Although we believe that our approach could lead to a definite improvement of combined events scoring, we decided to focus on running, since this is an exercise enjoyed by millions of people, rather than a handpicked elite.

Most recreational runners have great difficulty when comparing performances from one event to another. This becomes even more arduous when it comes to different individuals participating in different races. Our model allows to answer the question of performance comparison in a precise, quantitative, way. It can provide a universal scoring, expressed as a number of points, for any running distance. In order to give runners a friendly 'handle' we have introduced the term TraxScore. This allows comparisons between performances of a given runner in different events but also comparisons between different runners.

The usefulness of our approach is multiple. Knowing his TraxScore a runner can adjust his objectives over various distances in a more precise way. A universal score allows also to monitor in a precise way the runner's progress. Concerning race organizers, the use of the TraxScore would allow to group the participants in a race into corrals of runners of roughly the same value and make possible well-ordered staggered starts. Moreover, knowing one's level of ability should allow the runner to plan his training in a rational way, helping him to improve his performances while minimizing the injury risk.

The development of the TraxScore is part of an ambitious project of ours, the Human Performance Modelling. A first objective consists in making the TraxScore available to road races all over the world. Using race results we intend to refine the parametrization of our scoring formula. We will thus be able to provide a more accurate scoring for the various age groups for both sexes. Accounting for the altitude of the course, could be included in a future version of TraxScore. A more ambitious extension would be to provide scoring taking into account environmental factors, like temperature, humidity and weather in general. These improvements would make possible handicap races, both in real time and virtually. The existence of a top-quality TraxScore would assist organizers in their task and be a most useful guide to millions of runners worldwide.

Several directions of further research appear at this point. First, it would be easy, provided organized data exist, to apply our scoring approach to other locomotion-based disciplines like cycling or swimming. Second, it would be interesting by analyzing real-world data from various popular races to provide a feedback to the scoring proposed by World Athletics. Of course, we are aware that this is a tall order necessitating a substantial investment (not only) in time. Finally, since scoring tables are traditionaly related to combined events, it would be interesting to extend the treatment presented here to field events, reconnecting thus with the initial program of one of us, who in [Purdy 1972] proposed scoring tables with the decathlon in mind.

References

[Martens 1976] R. Martens, *Competition: In need of a theory*. In D. M. Landers (Ed.), Social problems in athletics. Urbana: University of Illinois Press (1976) 9.

[Golden 2004] M. Golden, *Sport in the Ancient World*, Routledge (London), 2004.

[Miller 2004] S.G. Miller, *Ancient Greek Athletics*, Yale Univ. Press (New Haven), 2004.

[Leonard 2016] J. Leonard, *The Value of Athletic Glory in Ancient Greece*, online at https://shorturl.at/bvJ15, 2016.

[Zarnowski 1989] F. Zarnowski, *The Decathlon*, Leisure Press (New York), 1989.

[Harder 2001] D. Harder, *Sports Comparisons*, Education Plus (Castro Valley CA) 2001.

[Grammaticos 2007] B. Grammaticos, *The physical basis of scoring the athletic performance*, New Stud. Athl. 22:3 (2007) 47.

[Grammaticos-Meloun-Purdy 2022] B. Grammaticos, J. Meloun and J.G. Purdy, *Distribution of performances and scoring in athletics*, Math. and Sports 3 (2022) 1.

[Purdy 1974-77] J.G. Purdy, *Computer generated track and field scoring tables*, in three parts: I. Historical development, II. Theoretical foundation and development of a model, III. Model evaluation and analysis, Med Sci. Sports 6 (1974) 287, 7 (1975) 111, 9 (1977) 212.

[Trkal 2003] V. Trkal, *The development of combined events scoring tables and implications for the training of decathletes*, New Stud. Athl. 18:4 (2003) 7.

[Purdy 1972] J.G. Purdy, *The application of computers to model physiological effort in scoring tables for track and field*, PhD thesis, Stanford Univ. 1972.

[Spiriev 2022] B. Spiriev, *IAAF Scoring Tables of Athletics*, World Athletics (Monaco), 2022.

[Wikipedia] Wikipedia, *Ultramarathon*, online at https://en.wikipedia.org/wiki/Ultramarathon.

[Grammaticos-Charon 2014] B. Grammaticos and Y. Charon, *Comparing the best athletic performances of the two sexes*, New Stud. Athl. 29:4 (2014) 37.

[Purdy 1974] J.G. Purdy, *Least square model for the running curve*, Res. Q. Exerc. Sport 45 (1974) 224.

[Grammaticos 2009] B. Grammaticos, *Scoring the athletic performance for age groups*, New Stud. Athl. 24:3 (2009) 63.

[Grammaticos 2020] B. Grammaticos, *On the Rise and Fall of athletic performances*, online at https://rethinkingathletics.blogspot.com/2020/06/on-rise-fall-of-athletic-performances.html