Mathematics and Sports

Decathlon Rules: An axiomatic approach

Guillaume Chèze[1,*]

[1]Institut de Mathématiques de Toulouse, UMR 5219
Université de Toulouse ; CNRS
UPS IMT, 118 route de Narbonne
F-31062 Toulouse Cedex 9, France.

[*]Corresponding Author E-mail:
guillaume.cheze@math.univ-toulouse.fr

Abstract
It has been shown that the current IAAF decathlon scoring tables involve certain biases. In the current scoring system, scoring rules depend on specific parameters. These parameters are difficult to define. Indeed, different arguments give different values for the parameters. Then, an obvious question arises: are these parameters mandatory to obtain a fair ranking between the athletes? This article will show that the answer to this question is no.
It will equally prove that if we want a ranking method satisfying four natural fairness axioms then this ranking method is unique. Moreover, this ranking method is easy to apply.

Keywords: Decathlon, rules, fairness, ranking, axioms

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1 Introduction

Combined track and field events have a long history. Indeed, the pentathlon was already practiced during antiquity. The events are well documented: there were the long jump, javelin throw, discus throw, the stadion foot race and wrestling. Unfortunately, the method used to determine the winner is not clear and nowadays several hypotheses exist. However, some elements allow us to deduce that a victorious athlete in the first three events was declared the overall winner, see e.g. Young (2004). This implies that the final result of the competition depended on the rankings in the different events and not on the performance in each event.

Nowadays, the most popular combined events in athletics are the decathlon and the heptathlon. Remember that the decathlon consists of ten track and field events which are divided in a two-day competition. Day 1: 100 metres, long jump, shot put, and high jump. Day 2: 110 metres hurdles, discuss throw, pole vault, javelin throw, and 1500 metres.

Since the beginning of the 20th century, the strategy used to rank the athletes has been based on the performance and not on the ranking in each event. More precisely, we associate to each performance a number of points: after the ten events we sum the points and then the winner is the athlete with the maximum number of points. Thus, if an athlete wins the first six events, we cannot declare him or her as the overall winner. Indeed, suppose that during the first six events an athlete A, wins 800 points in each event and an athlete B, wins 790 points in each event. Suppose also that during the four last events A obtains 790 in each event and B obtains 830 in each event. Then the final score of A is: $800 \times 6 + 790 \times 4 = 7960$ points, and the final score of B is: $790 \times 6 + 830 \times 4 = 8060$ points. Therefore, A is not the winner of the decathlon even if A was the best during the first 6 events.

The approach used today is thus based on the comparison of the performances and not on the comparison of the rankings in the different events. With the modern approach, competitors are more motivated to do their best in all events. However, this leads to a serious problem: How can we compare fairly the performance in different events? Indeed, how can we compare the result of a race expressed in seconds and the result of a jump expressed in centimetres?

The IAAF decathlon scoring tables use formulas which convert performances into a number of points. Several authors have shown that these formulas are not fair and that some events are favored, see e.g. Tidow (1989, 2000); Westera (2006, 2011). Obviously, the same kind of criticism can be made for the heptathlon, see Gassmann et al. (2016).

Westera has proposed a method which considers all events in the same way, see Westera (2006). This article, will attempt to take Westera's work a step further. We propose a new formula which transforms performances into points. Our formula is a simplification of Westera's. The paper will then go on to show how to aggregate all the scores obtained by each competitor in order to get a fair ranking. Indeed, usually the way to find the overall winner is to consider the mean of the scores obtained by each competitor. If we denote by s_i the score obtained after the i-th event, the mean of the scores is $\frac{s_1+s_2+\cdots+s_{10}}{10}$. More precisely, this mean is called the arithmetic mean. Nowadays, the winner is the athlete with the highest arithmetic mean. We remark that the highest arithmetic mean corresponds to the highest sum of the scores. Thus, in actual fact, we compute the sum of the scores. However, there exist different notions of mean. The Pythagorean means are: the arithmetic mean, the geometric mean and the harmonic mean. The geometric mean of ten numbers s_1, s_2, $\dots$, s_{10} is $(s_1 \times s_2 \times \cdots s_{10})^{1/10}$ and the harmonic mean is $\frac{10}{1/s_1+1/s_2+\cdots+1/s_{10}}$. For example, the geometric mean is used to compute the Human Development Index and the harmonic mean is used to compute the average speed when during a trip the speed is not the same during the outward journey and the return journey. This raises the following question: Why don't we use another mean: the geometric mean or the harmonic mean?
Indeed, it has not been proved that computing the sum of the score is the fairest way to find the overall winner of a decathlon. This article is going to show that the geometric mean is better suited for our problem. Roughly speaking, we prove that **it is better to consider the product $s_1 \times s_2 \times \cdots \times s_{10}$ of the scores rather than the sum $s_1 + s_2 + \cdots + s_{10}$**.

It is pretty straightforward why it is natural to consider the product instead of the sum of the scores. Indeed, suppose that we consider an imaginary competition with only two events. We also suppose that each event allocates 100 points at the most. Now, consider a competitor with the following scores (40,100). This means that A wins 40 points in the first event and 100 points in the second. Consider also another competitor B who has reached the scores (65,65). If we consider the sum of the scores then A wins the competition because A has 140 points and B has 130 points. However, it seems that A is a specialist of the second event and not very good at the first event. Competitor B seems to be more versatile. The product of the scores gives 4225 points for B and 4000 points for A. Thus by considering the product, B becomes the winner of the competition. This shows that the product of the scores allows to avoid that a specialist wins a competition of combined events. In this article, our *hypothesis* is that a decathlon tries to reward the more versatile athlete. Thus, this first example suggests that the product of the scores can be an alternative to the product of the scores.

We remark that the use of a product instead of a sum in order to get a fair ranking is already implemented in combined climbing competitions, see IFS (2018)[1]. Indeed, in the next Summer Olympics, climbing will be an Olympic sport. Competitors will take part in three disciplines (lead, speed and bouldering). Each discipline will give a ranking between the competitors and the score given after an event to a competitor will correspond to their rank. Thus in this situation, competitors will want to obtain a small score. Then, at the end of the competition, for each competitor, the product of the scores will be computed. The overall winner will be the competitor with the smallest product. For example, if a competitor is in second position in two events and is in the eighth position in the third event, then the product is: $2 \times 2 \times 8 = 32$ and this competitor gets 32 points. If another competitor is in the third position in all events then the product is: $3 \times 3 \times 3 = 27$ and this competitor only gets 27 points. Then, with the rule used for this kind of competition, this last competitor is better than the previous one because his final score is smaller.
Therefore, the ranking used in the Olympic climbing event is based on a product and not on a sum. However, the rule used in this kind of competition is based on the ranking obtained at the end of each event and is not based on the performance of each competitor.

In our study of decathlon rules, we are going to keep the spirit of the current rules. This means that we are going to consider a ranking based on the performance of each athlete and not based on the ranking at the end of each event.

We are going to study separately two problems:

1. How can we associate fairly a score, based on the performance, to each event?

2. How can we find a fair final ranking based on the score of each event?

It seems that the study of decathlon rules has never been questioned using these two paradigms. We think that the second question has never been studied because, usually, we always add points and the sum gives the ranking. This is the reason why this question was considered trivial. However, such is not the case. Indeed, as shown in the previous example, the product of the score gives another possibility of ranking. The fairness of the scoring rule for each event and the fairness of the final ranking are two different matters.

[1] https://www.ifsc-climbing.org/images/World$_{c}ompetitions/Event_{r}egulations/IFSC-Rules_{2}018_{V}1.5.pdf$

In this article, we are going to use an axiomatic approach in order to answer these two questions. More precisely, we are going to define axioms, i.e. basic rules, that we want to satisfy. These basic rules will keep the spirit of the existing scoring rules and their improvements proposed by Westerra. Then **we will show that there exists only one possible method of ranking satisfying these axioms.** We do not claim that the method proposed in this article is the better one. We prove that if we want to have a ranking method satisfying these properties then this method is unique.

Of course our basic principles have some athletic and philosophic motivations. As stated previously, we *suppose* that a decathlon tries to reward the more versatile athlete. Our work does not try to give a formal definition of what is the more versatile athlete. We are going to define axioms which are coherent with this goal and then from these axioms we are going to define a ranking.

For example, one of our principles is the “finisher axiom".
This axiom states that an athlete with a score equal to zero in an event cannot be considered as better than an athlete having only positive scores. This axiom does not imply that an athlete with a very low score during an event will not be able to win the decathlon.
The motivation for this axiom is the following: When we consider a competition with combined events, we want the athletes to finish *all* events. Athletes should not have the possibility to avoid an event. If the system used for ranking allows this possibility then several questions arise: How many events can an athlete avoid? Is it fair to compare two athletes when the former one has competed in ten events and the latter just nine or eight? Furthermore, if an athlete avoids some events during a decathlon e.g. shot put, discus and javelin throw, then the throw events are not taken into account. Thus, we cannot say that this athlete is versatile.

The study of ranking rules is not new. Mathematicians and economists have found several types of results in regards to this problem. Our result is in the spirit of classical results in social choice theory. In our situation, the proof of our theorem follows easily from our four axioms.

The structure of this article is thus simple: First, we will analyse how to associate fairly a score to a performance. Second, we will show how to obtain a fair ranking from scores. Lastly, we will give several examples in order to show the impact of the proposed ranking method.

2 Fair scoring rules

2.1 State of the art

In the document “IAAF scoring tables for combined events", see IAAF (2001), the history of the tables is presented. Different systems have been used since the 19th century. We recall here just some steps in order to introduce our scoring rule.
The first scoring tables were linear. This means that the relation between the performance and the number of points is of the following kind:

$$S = A \times (P - B),$$

where S is the score (the number of points), P is the performance of the athlete (for example the height in centimetres for the high jump). The numbers A and B were chosen as follows: B represents the performance giving 0 points and A is computed in order to give 1000 points to an identified top performance, for example the world record. For the running events the factor $(P - B)$ becomes $(B - P)$. Indeed, a good performance for these events corresponds to a shorter time.

The problem with this method is the following: a 1 cm improvement in a high jump performance increases the number of points in the same way no matter what the performance is. However, if you jump 75 cm and then 76 cm it is not the same performance as if you jumped 220 cm and then 221 cm. Thus, new scoring rules have been introduced: progressive scoring rules. These rules are based on formulas of the following kind:

$$S = A \times (P - B)^C, \text{ with } C > 1.$$

We remark that a new constant $C > 1$ has been introduced. To each event are thus associated three parameters A, B and C.
With this kind of formula, we get more points if we improve our performance from 220 cm to 221 cm than from 75 cm to 76 cm.

The current decathlon scoring tables use these kinds of progressive formulas and they have been used without modification since the 1980s.
When these rules were established in March 1983, nine points were accepted as basic principles for a new set of tables, see (IAAF, 2001, page 18):

1. The new set of tables should be used for combined events only.

2. Results in various events should, as far as possible, yield about the same number of points if the results are comparable as to quality and difficulty.

3. The new tables should be either:

   (a) a modification of the existing ones

   (b) a straight line in all events

   (c) slightly progressive tables in all events.

4. It must be possible to use the scoring tables for beginners, juniors, and top athletes as well.

5. There will be a special scoring table for men and another table for women.

6. All the new versions of the scoring tables should be based on the statistical data for the combined events by paying due regard to the statistical data for performances by single event athletes.

7. The new tables should be applicable now and for the future.

8. It is desirable without creating other problems, that the total scores using the new tables for the top world class athletes should remain approximately the same. That is about 8500 points for the decathlon and about 6500 points for the heptathlon.

9. As far as possible the new tables must insure that a specialist in one event cannot overcome performances in the other events.

However, the current system does not respect the second point. Indeed, if you equal the men's long jump world record (8.95m, Mike Powell, 30 August 1991) you get 1312 points and if you equal the men's 1500 metres world record (3 min 26 s, Hicham El Guerrouj, 14 July 1998) you win 1218 points. This means that the same outstanding performance in different events does not give the same number of points.
Furthermore, this difference $1318-1218=94$ points is not negligible. For example, during the IAAF world championship in London (2017), the difference of the number of points between the silver medal (Freimuth, 8564 points) and the bronze medal (Kazmirek, 8488 points) was just 76 points.
Moreover, 9000 points is an outstanding performance during a decathlon. Only three men have achieved such a performance. Thus 900 points is the average value for an event during an outstanding decathlon. Therefore, if we think about the 94 points as an error of measurement related to the IAAF method, then this error of measurement corresponds to more than 10% of the score obtained during an event. Thus, a difference of 94 points is not negligible.

Westera has suggested, in Westera (2006), a method which avoids this problem. Instead of considering a performance P we consider a normalized performance P_N defined as follows:

$$P_N = \frac{P-P_0}{P_1-P_0}.$$

where P_1 represents a high level performance (for example the world record) and P_0 a low-level performance. Thus if $P=P_1$, then $P_N=1$ and if $P=P_0$, then $P_N=0$.
For running events the performance will be the speed, or the inverse of the time. Thus, for running events if P is the time in seconds, then P_N is defined as follows:

$$P_N = \frac{\frac{1}{P}-\frac{1}{P_0}}{\frac{1}{P_1}-\frac{1}{P_0}}.$$

The idea of the normalized performance is thus to compare a performance with a standard. If an athlete jumps a height corresponding to 80% of the world record, we can consider that this performance is equivalent to a long jump corresponding to 80% of the length of the world record. This leads to the following formula:

$$S = A \times P_N^C.$$

In order to treat all events in the same way, Westera has proposed to take the same values A and C for all events. He has suggested: $A=863.9$ and $C=1.479$. These values were chosen in order to get scores relatively close to the ones given by the current scoring tables.

We notice that by construction, this method satisfies the second point of the nine basic principles accepted by IAAF.

2.2 A new formula

Westera has suggested a method that consider events in the same way, but a question still remains: *How do we choose P_0 and P_1 for each event?*

In the previous discussion we have written that we can take P_1 as the world record but we can also take P_1 as the average of the 100 best performances of all time. If we consider the last proposition, do we consider the 100 best performances obtained during a decathlon? As already remarked by Günter Tidow, see Tidow (1989), this raises the question of considering an absolute or a relative basis for scoring rules. If we consider a relative system (this means that we consider uniquely the performance obtained during a decathlon) then we take into account that after two days of competitions it is not fair to compare a performance during a decathlon to performances of single event specialists. Nevertheless, if a decathlon is a test of all-round ability then it seems natural to compare the performance of an athlete with the performance of a specialist. The choice of P_1 seems thus to be an unsolvable problem. There do not exist scientific nor philosophic arguments for choosing P_1.

The situation for P_0 seems easier. For example, for running events we have to choose a reference for low level performances. Nowadays this reference is 18 seconds for 100 m. Thus, 18 seconds gives zero points. Why 18 and not 19? If somebody walks slowly and needs 50 seconds for 100 m he will also get zero points. It is not fair and moreover we can always imagine a worse scenario. Here, we thus suggest to take $P_0 = \infty$, then $1/P_0 = 0$. This just means that the worst performance corresponds to a situation where the speed is equal to 0. This results, for running events, with:

$$\tilde{P}_N = \frac{P_1}{P},$$

where P is the time in seconds.

For the other events (jumps and throws), if we define a low level performance we can always imagine a lower performance except if the low level performance P_0 is fixed at 0. This results in:

$$\tilde{P}_N = \frac{P}{P_1}.$$

This leads us to suggest the following kind of scoring rules:

$$S = A \times \tilde{P}_N^C.$$

We can write, for all events except running, this formula as:

$$S = \frac{A \times P^C}{P_1^C}, \tag{Score 1}$$

and for running events we get:

$$S = \frac{A \times P_1^C}{P^C}. \tag{Score 2}$$

The parameter P_0 has disappeared and we still do not know how to fix A, P_1 and C. However, in the following part we will prove that these values are unnecessary to get a ranking between the athletes.

Once again, we notice that, by construction this scoring rule satisfied the second point of the nine basic principles accepted by IAAF.

3 Fair ranking based on scores

At the end of the competition, when we have all the scores of all competitors, we have to rank the athletes. The problem is thus the following: consider the scores obtained by two athletes $At_{\underline{s}}$ and $At_{\underline{t}}$, respectively $\underline{s} = (s_1, \dots, s_{10})$ and $\underline{t} = (t_1, \dots, t_{10})$, how can we rank $At_{\underline{s}}$ and $At_{\underline{t}}$ thanks to $\underline{s}$ and $\underline{t}$?
This means that we are looking for a ranking.

Definition 1. A ranking $\succ$ is a relation between athletes. In the following:
$At_{\underline{s}} \succ At_{\underline{t}}$ means $At_{\underline{s}}$ is strictly better than $At_{\underline{t}}$,
$At_{\underline{s}} \succeq At_{\underline{t}}$ means $At_{\underline{s}}$ is better or equal to $At_{\underline{t}}$,
$At_{\underline{s}} \approx At_{\underline{t}}$ means $At_{\underline{s}}$ is equal to $At_{\underline{t}}$, there is a tie.

In the following, if the scores associated to athlete $At_{\underline{s}}$ are $\underline{s} = (s_1, \dots, s_{10})$ and to athlete $At_{\underline{t}}$ are $\underline{t} = (t_1, \dots, t_{10})$ then we will write without distinction $\underline{s} \succ \underline{t}$ or $At_{\underline{s}} \succ At_{\underline{t}}$. In the same way, we will write without distinction $\underline{s} \approx \underline{t}$ or $At_{\underline{s}} \approx At_{\underline{t}}$.

3.1 One event is sufficient to give the victory

A natural axiom satisfied by the current rule is **“one event is enough to give the victory".** In the following we will denote this axiom by **1EV** (one event victory).

Definition 2. A ranking satisfies the rule “one event is enough to give the victory" (1EV) if it satisfies the following:
If $s_1 = t_1, s_2 = t_2, \dots, s_9 = t_9$ and $s_{10} > t_{10}$ then $\underline{s} \succ \underline{t}$.
Reciprocally, if p is a performance and $(p, p, \dots, p, s_{10}) \succ (p, p, \dots, p, t_{10})$ then $s_{10} > t_{10}$.

This axiom is natural and means that if two athletes $At_{\underline{s}}$ and $At_{\underline{t}}$ make the exact same performance during the first nine events and $At_{\underline{s}}$ is better than $At_{\underline{t}}$ in the last event then $At_{\underline{s}}$ is better than $At_{\underline{t}}$ in the overall competition.

3.2 The equal weight axiom

The second natural and basic rule that we want to satisfy for a ranking is the **equal weight axiom**. The idea is the following: the name of the event where competitors win their points is not important, what really matters is the number of points. Each event must weigh the same in the final ranking.

Definition 3. A ranking satisfies the *equal weight axiom* if it satisfies the following rule:
if $\underline{t}$ is obtained from $\underline{s}$ by exchanging coordinates then $\underline{t} \approx \underline{s}$

This means that the order of the score is not important, for example:

$$(1,2,3,4,5,6,7,8,9,10) \approx (10,3,5,2,7,6,1,4,9,8).$$

The equal weight axiom is satisfied by the current rule. Indeed, the sum of the scores is independent of the order of the different scores.

3.3 The finisher axiom

Definition 4. Let $z_{\underline{s}}$ be the number of scores equal to zero in $\underline{s}$.
A ranking which satisfies the *finisher axiom* means that: If $z_{\underline{s}} < z_{\underline{t}}$ then $\underline{s} \succ \underline{t}$.

This axiom means, for example, that an athlete with one score equal to zero cannot be considered better than another athlete with all his scores positive. This axiom is not theoretically satisfied by the current rule. However, in practice it is impossible to win a decathlon when one of the score is equal to zero.

In regards with this axiom one can argue: “An international decathlete failing to complete one event ends up with a lower rank than a novice who completes ten events to a poor level of performance. This is a problem. Even if an athlete has a score equal to zero in an event then he or she could get a good rank."
However, if we accept this last statement then it implies that an athlete can avoid an event. This leads to the question: how many events an athlete can avoid? Just one, two... In this situation, the competition will not reward the more versatile athlete and it is in contradiction with our hypothesis.

3.4 The scale independence axiom

Now, we are going to introduce a new axiom in order to avoid some difficulties raised by the choice of parameters A and P_1. The problem with these parameters is the following: if we change them then we change the scores and thus probably the ranking. For example, if we use (Score 1) with $P_1 = P_W$ where P_W is the world record of the considered event or with P_a where P_a is the average of the 100 best performances of all time then we get two different scoring rules S_W and S_a where:

$$S_W = \frac{A \times P^C}{P_W^C}; \quad S_a = \frac{A \times P^C}{P_a^C}.$$

It follows

$$(\star) \quad S_W = \alpha \times S_a \text{ where } \alpha = \frac{P_a^C}{P_W^C}.$$

Thus S_W and S_a are proportional.

Now, we consider

$$\underline{s}_w = (S_{w,1}, \dots, S_{w,10}) \text{ and } \underline{t}_w = (T_{w,1}, \dots, T_{w,10})$$

the 10 scores obtained in each event by two athletes, when the parameter P_1 is the world record, and

$$\underline{s}_a = (S_{a,1}, \dots, S_{a,10}), \quad \underline{t}_a = (T_{a,1}, \dots, T_{a,10})$$

the 10 scores obtained in each event by the same athletes, but when the parameter P_1 is the average of the 100 best performances of all time.
By $(\star)$, we know that for the first event and thus the first score there exists a coefficient of proportionality α_1 satisfying $S_{w,1} = \alpha_1 \times S_{a,1}$. The other events give other coefficients of proportionality $\alpha_2, \dots, \alpha_{10}$. Then, we can write

$$(\star\star) \quad \underline{s}_w = (S_{w,1}, \dots, S_{w,10}) = (\alpha_1 \times S_{a,1}, \dots, \alpha_{10} \times S_{a,10}).$$

Now, in order to simplify the notations we write

$$\underline{\alpha} = (\alpha_1, \dots, \alpha_{10}) \text{ and } \underline{\alpha}.\underline{s}_a = (\alpha_1 \times S_{a,1}, \dots, \alpha_{10} \times S_{a,10}).$$

With these notations, $(\star\star)$ can be written as

$$\underline{s}_w = \underline{\alpha} \cdot \underline{s}_a.$$

Therefore, if we want to have a ranking independent of the choice of the parameter P_1 then we must have

$$\underline{s}_a \succ \underline{t}_a \text{ if and only if } \underline{s}_w \succ \underline{t}_w.$$

However, as $\underline{s}_w = \underline{\alpha} \cdot \underline{s}_a$ and in the same way $\underline{t}_w = \underline{\alpha} \cdot \underline{t}_a$ this gives:

$$\boxed{\underline{s}_a \succ \underline{t}_a \text{ if and only if } \underline{\alpha} \cdot \underline{s}_a \succ \underline{\alpha} \cdot \underline{t}_a.}$$

If we study the situation for the choice of parameter A, we get the same kind of condition.

Since it is impossible to objectively decide a good value for P_1 and A, we can look for ranking independent of these parameters. The previous discussion shows that the rank has to satisfy the next axiom: **scale independence**.

Definition 5. We say that a ranking $\succ$ is *scale independent* if

$$\underline{s} \succeq \underline{t} \iff \underline{\alpha}.\underline{s} \succeq \underline{\alpha}.\underline{t},$$

where $\underline{\alpha}.\underline{s} = (\alpha_1 \times s_1, \alpha_2 \times s_2, \dots, \alpha_{10} \times s_{10})$, with $\underline{\alpha} = (\alpha_1, \dots, \alpha_{10})$, $\underline{s} = (s_1, \dots, s_{10})$ and $\alpha_i > 0$ for all i.

The scale independence axiom implies that the ranking is independent of P_1. However, the choice of P_1 is not just a problem of fairness. Indeed, whatever the choice that we make for parameter P_1, it will change in the future. New records will be set. The scale independence axiom also says that it will be unnecessary to change the rule if the high level performance P_1 changes. Therefore, this last axiom avoids a lot of practical problems. Furthermore, it means that the seventh basic principle accepted by the IAAF: “The new tables should be applicable now and for the future.", see Section 2, is satisfied.

3.5 The Nash relation

The question now is: *Is there a ranking satisfying the finisher axiom, the equal weight axiom, 1EV and scale independence?*

The following ranking $\underset{Nash}{\succ}$ gives an answer to this question.

Definition 6. We set $\underline{s} \underset{Nash}{\succ} \underline{t}$ when $z_{\underline{s}} < z_{\underline{t}}$ or when $z_{\underline{s}} = z_{\underline{t}}$ and

$$\prod_{s_i \neq 0} s_i > \prod_{t_i \neq 0} t_i.$$

We set $\underline{s} \underset{Nash}{\approx} \underline{t}$ when $z_{\underline{s}} = z_{\underline{t}}$ and $\prod_{s_i \neq 0} s_i = \prod_{t_i \neq 0} t_i$.

Example 7. With this definition we have

$$(801, 802, \dots, 810) \underset{Nash}{\succ} (801, 801, \dots, 801),$$

because $801 \times 802 \times \cdots \times 810 > 801 \times 801 \times \cdots \times 801$.
We have chosen these kinds of scores $801, \dots, 810$ because this corresponds to a good score in an event with the current system.
Furthermore, if we set $\underline{s} = (801, 802, \dots, 810)$ and $\underline{t} = (900, 900, \dots, 900, 0)$, then we have $\underline{s} \underset{Nash}{\succ} \underline{t}$. Indeed, the number of scores equal to zero in $\underline{t}$ is equal to 1. Thus $z_{\underline{t}} = 1$. As in $\underline{s}$ all coefficients are positive we have $z_{\underline{s}} = 0$ and then we deduce $\underline{s} \underset{Nash}{\succ} \underline{t}$.

We call this relation the Nash relation because it is inspired by the work of John Forbes Nash who studied this kind of relation, see Nash (1950).

The Nash relation uses the geometric mean and not the arithmetic mean in order to compare two series. We recall that the geometric mean of n numbers is $\sqrt[n]{a_1 \times \cdots \times a_n}$. Here the n-th root of the product is not taken into account because we just want to compare two numbers and $x > y \iff \sqrt[n]{x} > \sqrt[n]{y}$.

The Nash relation satisfies the finisher axiom. Indeed, by definition we have $\underline{s} \underset{Nash}{\succ} \underline{t}$ when $z_{\underline{s}} < z_{\underline{t}}$.

The Nash relation satisfies the equal weight axiom. Indeed, a product is independent of the order of the different factors.

The Nash relation satisfies 1EV because if $s_1 = t_1, \dots, s_9 = t_9$ and $s_{10} > t_{10}$ then $\prod_{s_i \neq 0} s_i > \prod_{t_i \neq 0} t_j$ and $\underline{s} \underset{Nash}{\succ} \underline{t}$.

The Nash relation is also scale independent. For example if we multiply by 2 the score of the first event (100 m) and by 3 the score in the last event (1500m) then this situation corresponds to $\underline{\alpha} = (2, 1, 1, 1, \dots, 1, 3)$. Suppose that $\underline{s} \underset{Nash}{\succ} \underline{t}$ and $s_i \neq 0$, $t_i \neq 0$, for $i = 1, \dots, 10$. Then we have

$$s_1 \times s_2 \times \cdots \times s_{10} > t_1 \times t_2 \times \cdots \times t_{10}$$

$$\Updownarrow$$

$$2s_1 \times s_2 \times \cdots \times 3s_{10} > 2t_1 \times t_2 \times \cdots \times 3t_{10}$$

$$\Updownarrow$$

$$\underline{\alpha}.\underline{s} \underset{Nash}{\succ} \underline{\alpha}.\underline{t}.$$

Actually, the Nash relation is the only relation satisfying our four axioms. In order to state this characterization, we have to define the following notion:

Definition 8. We say that two rankings $\succ_1$ and $\succ_2$ coincide when:

$$\underline{s} \succ_1 \underline{t} \iff \underline{s} \succ_2 \underline{t}.$$

This definition says that if $\succ_1$ and $\succ_2$ are two rankings defined in two different ways but give the same result, then we say that $\succ_1$ and $\succ_2$ coincide.

Example 9.

- If $\succ_1$ is defined as follows:
  $\underline{s} \succ_1 \underline{t}$ when $s_1 + s_2 + \cdots + s_{10} > t_1 + t_2 + \cdots + t_{10}$,
  and $\succ_2$ is defined in the following way :
  $\underline{s} \succ_2 \underline{t}$ when $\frac{s_1 + s_2 + \cdots + s_{10}}{10} > \frac{t_1 + t_2 + \cdots + t_{10}}{10}$,
  then $\succ_1$ and $\succ_2$ give the same ranking. Then $\succ_1$ and $\succ_2$ coincide.
- If $\succ_1$ is defined as follows:
  $\underline{s} \succ_1 \underline{t}$ when $s_1 \times s_2 \times \cdots \times s_{10} > t_1 \times t_2 \times \cdots \times t_{10}$,
  and $\succ_2$ is defined in the following way:
  $\underline{s} \succ_2 \underline{t}$ when $2 \times (s_1 \times s_2 \times \cdots \times s_{10}) > 2 \times (t_1 \times t_2 \times \cdots \times t_{10})$,
  then $\succ_1$ and $\succ_2$ give the same ranking. Then $\succ_1$ and $\succ_2$ coincide.
- Now, we define $\underset{deca}{\succ}$ in the following way:

– If $z_{\underline{s}} < z_{\underline{t}}$ then $\underline{s} \underset{deca}{\succ} \underline{t}$.

– If $z_{\underline{s}} = z_{\underline{t}}$ and

$$7670 \times \Big(\prod_{s_i \neq 0} s_i\Big)^{0.23} > 7670 \times \Big(\prod_{t_i \neq 0} t_i\Big)^{0.23}$$

then $\underline{s} \underset{deca}{\succ} \underline{t}$.

As the function $f(x) = 7670x^{0.23}$ is an increasing function we deduce that $\underset{Nash}{\succ}$ and $\underset{deca}{\succ}$ coincide.

The ranking $\underset{deca}{\succ}$ will be used when we study examples. Indeed, the number $7670 \times \Big(\prod_{i=1}^{10} s_i\Big)^{0.23}$ has approximately the same order of magnitude as the number of points obtained with the IAAF scoring method. Thus with $\underset{deca}{\succ}$ we get the same ranking as the one obtained with the Nash relation but the number of points given by the formula $7670 \times \Big(\prod_{i=1}^{10} s_i\Big)^{0.23}$ can be compared with the IAAF score.

In the same way, the formula $f(\underline{s}) = \alpha \times \Big(\prod_{s_i \neq 0} s_i\Big)^{\beta}$, where α and β are positive, gives a ranking. This ranking coincides with the one given by $\underset{Nash}{\succ}$. The choice of α and β in order to have the best approximation of the IAAF score with the function $f(\underline{s})$ would need a statistical study. In the following the relation $\underset{deca}{\succ}$ will be sufficient for our study.

We then have the following characterization:

Theorem 10 (Characterization of the Nash relation)**.**

- *The Nash relation satisfies the finisher axiom, the equal weight axiom, 1EV and scale independence.*

- *Conversely, if a relation defined on non-negative numbers satisfied the finisher axiom, the equal weight axiom, 1EV and is scale independent then it coincides with the Nash relation.*

Proof. We have already ascertained the first point. Thus, we only have to prove the second point.

We consider a relation $\succ$ defined on $\mathbb{R}_+^{10}$, where $\mathbb{R}_+ = [0; +\infty[$, which satisfies the finisher, equal weight, 1EV and scale independence axioms, and we are going to show that $\succ$ coincides with $\underset{Nash}{\succ}$.

We consider $(s_1, s_2, \dots, s_{10})$ and $(t_1, t_2, \dots, t_{10})$.
If $z_{\underline{s}} < z_{\underline{t}}$ then $\underline{s} \succ \underline{t}$, by the finisher axiom. Thus the relation $\succ$ coincides with the Nash relation.

Now we suppose $z_{\underline{s}} = z_{\underline{t}}$. In order to simplify the notation, we suppose $z_{\underline{s}} = z_{\underline{t}} = 0$, this means that all scores are positive. We set:

$$(s_1, s_2, \dots, s_{10}) \succ (t_1, t_2, \dots, t_{10})$$

and we want to prove $(s_1, s_2, \dots, s_{10}) \underset{Nash}{\succ} (t_1, t_2, \dots, t_{10})$.

By the scale invariance axiom with $\underline{\alpha} = \Big(\frac{1}{s_1}, \frac{1}{s_2}, \dots, \frac{1}{s_9}, \frac{1}{t_{10}}\Big)$, we have:

$$(s_1, s_2, \dots, s_{10}) \succ (t_1, t_2, \dots, t_{10}) \iff \Big(1, 1, \dots, 1, \frac{s_{10}}{t_{10}}\Big) \succ \Big(\frac{t_1}{s_1}, \frac{t_2}{s_2}, \dots, \frac{t_9}{s_9}, 1\Big).$$

Furthermore, the equal weight axiom gives:

$$\Big(1, 1, \dots, 1, \frac{s_{10}}{t_{10}}\Big) \succ \Big(\frac{t_1}{s_1}, \frac{t_2}{s_2}, \dots, \frac{t_9}{s_9}, 1\Big) \iff \Big(1, 1, \dots, 1, \frac{s_{10}}{t_{10}}\Big) \succ \Big(1, \frac{t_1}{s_1}, \frac{t_2}{s_2}, \dots, \frac{t_9}{s_9}\Big).$$

Now, the strategy is to obtain more coordinates equal to 1 in the vector $(1, t_1/s_1, \dots, t_9/s_9)$.
The scale independence axiom with $\underline{\alpha} = \Big(1, 1, \dots, 1, \frac{s_9}{t_9}\Big)$ gives:

$$\Big(1, 1, \dots, 1, \frac{s_{10}}{t_{10}}\Big) \succ \Big(1, \frac{t_1}{s_1}, \frac{t_2}{s_2}, \dots, \frac{t_9}{s_9}\Big) \iff \Big(1, 1, \dots, 1, \frac{s_9 s_{10}}{t_9 t_{10}}\Big) \succ \Big(1, \frac{t_1}{s_1}, \dots, \frac{t_8}{s_8}, 1\Big).$$

The equal weight axiom gives:

$$\left(1,1,\ldots,1,\frac{s_9s_{10}}{t_9t_{10}}\right)\succ\left(1,\frac{t_1}{s_1},\ldots,\frac{t_8}{s_8},1\right)\iff\left(1,1,\ldots,1,\frac{s_9s_{10}}{t_9t_{10}}\right)\succ\left(1,1,\frac{t_1}{s_1},\ldots,\frac{t_8}{s_8}\right).$$

The scale independence axiom with $\underline{\alpha}=\left(1,1,\ldots,1,\frac{s_8}{t_8}\right)$ gives:

$$\left(1,1,\ldots,1,\frac{s_9s_{10}}{t_9t_{10}}\right)\succ\left(1,1,\frac{t_1}{s_1},\ldots,\frac{t_8}{s_8}\right)\iff\left(1,1,\ldots,1,\frac{s_8s_9s_{10}}{t_8t_9t_{10}}\right)\succ\left(1,1,\frac{t_1}{s_1},\ldots,\frac{t_7}{s_7},1\right).$$

The equal weight axiom gives:

$$\left(1,1,\ldots,1,\frac{s_8s_9s_{10}}{t_8t_9t_{10}}\right)\succ\left(1,1,\frac{t_1}{s_1},\ldots,\frac{t_7}{s_7},1\right)\iff\left(1,1,\ldots,1,\frac{s_8s_9s_{10}}{t_8t_9t_{10}}\right)\succ\left(1,1,1,\frac{t_1}{s_1},\ldots,\frac{t_7}{s_7}\right).$$

We continue with the same process and we get:

$$\left(1,\ldots,1,\frac{s_2s_3\cdots s_{10}}{t_2t_3\cdots t_{10}}\right)\succ\left(1,\ldots,1,\frac{t_1}{s_1}\right).$$

Now, the 1EV axiom gives:

$$\left(1,\ldots,1,\frac{s_2s_3\cdots s_{10}}{t_2t_3\cdots t_{10}}\right)\succ\left(1,\ldots,1,\frac{t_1}{s_1}\right)\iff\frac{s_2s_3\cdots s_{10}}{t_2t_3\cdots t_{10}}>\frac{t_1}{s_1}\iff\prod_{i=1}^{10}s_i>\prod_{i=1}^{10}t_i.$$

In conclusion, we have:

$$(s_1,s_2,\ldots,s_{10})\succ(t_1,t_2,\ldots,t_{10})\iff\prod_{i=1}^{10}s_i>\prod_{i=1}^{10}t_i\iff\underline{s}\underset{Nash}{\succ}\underline{t}.$$

Thus, if a relation defined on non-negative numbers satisfies our four axioms then it coincides with the Nash relation. □

Similar characterizations of the Nash relation with other axioms already exist, see Moulin (1988). However, in these characterizations the Nash relation is only used and proved on positive numbers. Here, we consider non-negative numbers because it is possible to have a performance equal to zero.

3.6 Application

We apply the previous theorem to the decathlon when scoring rules defined by formulas (Score 1) and (Score 2) are used.

Consider two athletes $At_{\underline{s}}$ and $At_{\underline{t}}$ with associated scores $(s_1,\ldots,s_{10})$ and $(t_1,\ldots,t_{10})$. The first score corresponds to the first event (100 m) and the last score to the last event (1500 m). We thus have

$$s_1=\frac{A\times P^C_{1,100}}{P^C_{\underline{s},100}},s_2=\frac{A\times P^C_{\underline{s},L}}{P^C_{1,L}},\ldots,s_{10}=\frac{A\times P^C_{1,1500}}{P^C_{\underline{s},1500}},$$

where $P_{1,100}$ is the high level performance chosen for the first event (100 m), $P_{1,L}$ the high level performance chosen for the second event (long jump),..., and $P_{1,1500}$ the high level performance chosen for the last event (1500 m). In the same way $P_{\underline{s},100}$ is the performance of the athlete $At_{\underline{s}}$ in the first event, etc.

Now, suppose that $\succ$ is a ranking defined on the scores which satisfies our four axioms, then by Theorem 10 this ranking coincides with the Nash relation. Thus we have the following equivalences:

$$\underline{s}\succ\underline{t}\iff z_{\underline{s}}<z_{\underline{t}}\text{ or }z_{\underline{s}}=z_{\underline{t}}\text{ and }\prod_{s_i\neq0}s_i>\prod_{t_i\neq0}t_i.$$

In order to simplify the notation we suppose $z_{\underline{s}}=z_{\underline{t}}=0$, this gives

$$\underline{s}\succ\underline{t}\text{ and }z_{\underline{s}}=z_{\underline{t}}=0$$

$$\Updownarrow$$

$$\frac{A\times P^C_{1,100}}{P^C_{\underline{s},100}}\times\frac{A\times P^C_{\underline{s},L}}{P^C_{1,L}}\times\cdots\times\frac{A\times P^C_{1,1500}}{P^C_{\underline{s},1500}}>\frac{A\times P^C_{1,100}}{P^C_{\underline{t},100}}\times\frac{P^C_{\underline{t},L}}{P^C_{1,L}}\times\cdots\times\frac{A\times P^C_{1,1500}}{P^C_{\underline{t},1500}}.$$

We can simplify the right hand side and the left hand side by dividing by A, $P_{1,100}^C, \dots, P_{1,1500}^C$ this gives:

$$\underline{s} \succ \underline{t} \text{ and } z_{\underline{s}} = z_{\underline{t}} = 0$$
$$\Updownarrow$$
$$\frac{P_{\underline{s},L}^C \times P_{\underline{s},H}^C \times P_{\underline{s},PV}^C \times P_{\underline{s},JT}^C \times P_{\underline{s},DT}^C \times P_{\underline{s},SP}^C}{P_{\underline{s},100}^C \times P_{\underline{s},400}^C \times P_{\underline{s},110}^C \times P_{\underline{s},1500}^C} > \frac{P_{\underline{t},L}^C \times P_{\underline{t},H}^C \times P_{\underline{t},PV}^C \times P_{\underline{t},JT}^C \times P_{\underline{t},DT} \times P_{\underline{t},SP}^C}{P_{\underline{t},100}^C \times P_{\underline{t},400}^C \times P_{\underline{t},110}^C \times P_{\underline{t},1500}^C}$$

where H means high jump, PV pole vault, JT javelin throw, DT discus throw and SP shot put.

Finally, we simplify the power C in the previous inequality. We get:

$$\underline{s} \succ \underline{t} \text{ and } z_{\underline{s}} = z_{\underline{t}} = 0$$
$$\Updownarrow$$
$$\frac{P_{\underline{s},L} \times P_{\underline{s},H} \times P_{\underline{s},PV} \times P_{\underline{s},JT} \times P_{\underline{s},DT} \times P_{\underline{s},SP}}{P_{\underline{s},100} \times P_{\underline{s},400} \times P_{\underline{s},110} \times P_{\underline{s},1500}} > \frac{P_{\underline{t},L} \times P_{\underline{t},H} \times P_{\underline{t},PV} \times P_{\underline{t},JT} \times P_{\underline{t},DT} \times P_{\underline{t},SP}}{P_{\underline{t},100} \times P_{\underline{t},400} \times P_{\underline{t},110} \times P_{\underline{t},1500}}.$$

We have thus proved the following:

Theorem 11. *Suppose that the score of the different events of the decathlon are given by formulas (Score 1) and (Score 2). Consider $\succ$ a relation on scores which satisfies finisher, equal weight, 1EV, and scale independence axioms. Then this relation follows from the final score given by the formula:*

$$S_F = \frac{P_{\underline{s},L} \times P_{\underline{s},H} \times P_{\underline{s},PV} \times P_{\underline{s},JT} \times P_{\underline{s},DT} \times P_{\underline{s},SP}}{P_{\underline{s},100} \times P_{\underline{s},400} \times P_{\underline{s},110} \times P_{\underline{s},1500}}.$$

Thus, if we want to use a ranking satisfying our four axioms then we necessarily must use a ranking which coincides with the one based on the final score computed by S_F. We notice that this final score is independent of the choice of the parameters A, C, $P_{1,100}$, $P_{1,L}$, $\dots$, $P_{1,1500}$.
Furthermore, we remark that in the numerator we have performances in centimetres (the greater, the better), and in the denominator performances in seconds (the shorter, the better).

4 Some examples

Let's now look into some examples and see what kind of results are given by this new ranking. We thus compare the ranking obtained with the current score and with the score S_F.

The values obtained with S_F are very different from the current scoring method. However, if we consider the score $S_{deca} = 7670 \times S_F^{0.23}$, then the current score and S_{deca} have the same order of magnitude. This means that in practice we can use the relation $\succ_{deca}$ previously defined, see Example 9. We have seen that this ranking coincides with the Nash ranking. Thus the rankings constructed from S_F and S_{deca} are the same and satisfy our four axioms.

In S_{deca} we have used the function $f(x) = 7670x^{0.23}$. The value 7670 and 0.23 have been chosen in order to have a simple relation: If $S_F = 2$ then $S_{deca} = 9000$ and if $S_F = 1.2$ then $S_{deca} = 8000$.
The previous choice is arbitrary. We can use other strictly increasing functions f in order to define a new score $S = f(S_F)$. The obtained ranking will coincide with the Nash ranking and $\succ_{deca}$.

Here, the scores S_F and S_{deca} are rounded with a sufficient precision in order to rank the athletes.

Table 1 shows the results of the 2017 IAAF World Championships. We give the score obtained by the first eight athletes using the current method and their ranking. In the last three columns we give the score obtained by these athletes with the proposed scores S_F, S_{deca} and their associated ranking.

The score S_F is computed with the time in seconds and the length and height in metres.
For example, K. Mayer's results were:

100 m: 10 s 70; Long Jump: 7.52 m; Shot Put: 15.72 m; High Jump: 2.08 m;
400 m: 48 s 26; 110 m hurdles: 13 s 75; Discus throw: 47.14 m; Pole vault: 5.10 m, Javelin throw: 66.10 m; 1 500m: 4 min 36 s 73.
This gives:

$$S_F = \frac{7.52 \times 15.72 \times 2.08 \times 47.14 \times 5.10 \times 66.10}{10.70 \times 48.26 \times 13.75 \times 276.73} = 1.988679948.$$

In Table 1, we notice that the proposed ranking based on the Nash relation with the formula S_F and the current ranking do not coincide.

Name	IAAF Points	Ranking with IAAF Points	Points with S_F	Points with S_{deca}	Ranking with S_F and S_{deca}
K. Mayer	8768	1	1.9886799	8983.9078	1
R. Freimuth	8564	2	1.7237276	8693.2696	2
K. Kazmirek	8488	3	1.5189938	8444.0976	4
J. Õiglane	8371	4	1.5677620	8505.6948	3
D. Warner	8309	5	1.2000365	7998.5283	8
O. Kasyanov	8234	6	1.2744814	8110.0227	7
K. Felix	8227	7	1.3922411	8276.5566	6
Adam Helcelet	8222	8	1.4589213	8366.0931	5

Table 1: Study of the different rankings for the IAAF World Championships London 2017.

The ranking of Kazmirek and Õiglane varies depending on the method. It can be perceived when we look at the “worst" event of these two athletes.
Õiglane was the 22-nd athlete of the 400 m, with 49,58s. The first athlete was Kazmirek with 47.19s. Thus, Õiglane’s performance represents $47.19/49.58 = 95.17\%$ of the best performance achieved during this event.
Kazmirek was the 21-st athlete of the shot put, with 13.78 m. The first athlete was Victor with 15.86m. Thus, Kazmirek’s performance represents $13.78/15.86 = 86,88\%$ of the best performance achieved during this event.
This means that Õiglane’s “bad" performance is better than Kazmirek’s. Thus, Õiglane seems more versatile.

In Table 2, we make the same comparison with the last five world records. In this situation we remark that T. Dvořák’s result was better than Šebrle’s and Eaton’s records if we use the Nash relation. However, Mayer’s IAAF world record (16/09/2018) remains the best result with the Nash relation.

Name	World Record with IAAF Points	Points with S_F	Points with S_{deca}
K. Mayer	9126	2.61256	9565.7884
A. Eaton	9045	2.03207	9028.6263
R. Šebrle	9026	2.29382	9283.7678
T. Dvořák	8994	2.39846	9379.5093
D. O’Brien	8891	2.12567	9122.6191

Table 2: Study of the different rankings for the world record

5 Conclusion

In this article we have suggested a method for ranking athletes at the end of a decathlon. Our approach has been divided in several steps. First, associate a fair score to each event. Here, fair means satisfying the second basic principle accepted by IAAF: results in various events should, as far as possible, yield about the same number of points if the results are comparable as to quality and difficulty.
This has led to a new scoring method using formulas (Score 1) and (Score 2). These formulas extend the work done by Westera. Indeed, the score is computed thanks to a normalized performance. In each event, if a performance corresponds to the same ratio of a chosen high level performance then the number of points obtained is the same. With our method for each event three parameters must be chosen. In Westera's approach, four were needed.
This proposed new rule is in the same spirit as the current rule but allows us to have the same construction for each event.

Secondly, we have proved that there exists a unique ranking method based on scores which satisfies four fairness axioms. Our first two axioms 1EV and equal weight are already satisfied by the current rule. The finisher axiom is satisfied in practice by the current rule and allows to satisfy the ninth basic principle accepted by IAAF. The fourth axiom, scale independence, has been introduced in order to have a ranking independent of the choice of the high level performance. Indeed, it seems impossible to define fairly what a high level reference for each event is. As a result we obtain a unique possible ranking method given by the Nash relation.
The advantage of this ranking is that it does not depend on arbitrary coefficients. Thus, this method does not depend on old or future performances. This means that it satisfies the seventh basic principle accepted by IAAF.

We remark that the proposed method is the same for men and women because the formula S_F is independent of parameters. However, for each event, formulas (Score 1) and (Score 2) use parameters P_1, A and C. Thus, if we want to give explicitly a score after an event we need to choose these parameters and they will be different for men and women. This means that our approach does not contradict the fifth basic principle accepted by IAAF: there will be a special scoring table for men and another for women.
Thus, if we want to give a score after each event then it is necessary to choose parameters and they will be different for men and women. However, these scores and parameters are not necessary to compute the final scores S_F which give the ranking. Only performances are used to compute the final scores S_F and to deduce the final ranking.

Lastly, the proposed ranking is easy to compute and if we want to satisfy our four natural axioms we have no other possible method.

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