Mathematics and Sports

Skating on circles: Using figure skating blades to apply circular geometry concepts

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Abstract
In this article, a secondary mathematics activity about a piece of sporting equipment, the figure skate blade, is presented. While a figure skate blade appears to be nearly flat, upon closer examination, it is a composition of circles' sectors. Using geometric and algebraic concepts, participants determined the radius of a figure skate blade. We conducted this activity with undergraduate students enrolled in a course on culturally-based perspectives of mathematics and science, a required course for students planning to become middle and high school teachers certified to teach mathematics and science. The participants used several different geometric ideas and models to solve the key question, and the benefits and drawbacks of these strategies are discussed. Since the activity involves participants' mathematizing the three-dimensional skate blade, the activity can be classified as a Model-Eliciting Activity (MEA).

Keywords: modeling, geometry, figure skating, education, MEA

Introduction

Figure skating is a popular sport for fans within the United States. According to Google Trends' analysis, figure skating was the most widely searched Winter Olympic sport on Google's website in 46 of the 50 states in the United States of America in the 12 months leading up to the Winter Olympic Games in 2018 (Ryerson, 2018). In 2018-19, the governing body of figure skating in the United States, US Figure Skating (2019), reported that over 200,000 members joined the organization, and this membership number was the highest annual membership total in the organization's history.

There are many opportunities to enjoy figure skating at a social and recreational level, as a participant in the sport or as a spectator. In the state of Maryland, where the mathematical activities reported in the article were conducted, there are 17 indoor ice-skating rinks and 7 seasonal outdoor ice-skating rinks. It is common for grade school students to have some exposure to ice skating through attendance at birthday parties, social events, and school field trips. As of 2019, there are 11 figure skating clubs in Maryland that are affiliated with US Figure Skating; one of these clubs, the Washington Figure Skating Club, has the largest membership total nationwide with over 1,100 members in each of the past five years, from 2013-14 through 2018-19 (US Figure Skating, 2019).

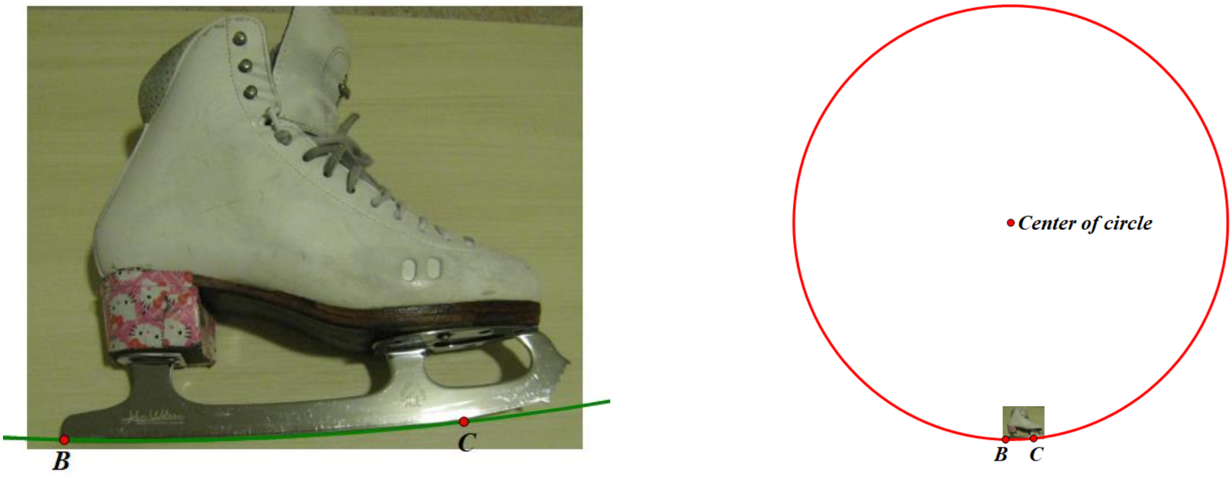
Figure skating can be used as a venue to discuss geometry and algebra in mathematical settings. For example, skaters' blade tracings that are formed on the ice can be described using parts of circles (Cheng, Berezovski & Talbert, 2019). Many of the bodily motions and poses within skaters' programs can be described by their angles (Berezovski & Cheng, 2016) and by transformations such as reflections, rotations, and translations (Cheng, 2019). Physical principles such as inertia and conservation of angular momentum while spinning and jumping inherent within figure skating can be described using algebraic equations (Cheng & Twillman, 2018).

Since buying used skates makes figure skating more financially affordable for families than buying new skates, figure skate shops commonly sell used skate boots and blades available for purchase – both in-person and online, and figure skating clubs often hold used skate sales as fundraisers. When shopping for used skates, it is relatively easy to discern a figure skate boot's condition; for maximum ankle support, skaters would can see that a boot has little or no remaining useful life when it shows creases in the side. But it is much harder to qualitatively determine the differences between skate blades, in order to distinguish whether the skate blade has been sharpened in a way that maintains its originally manufactured radius. If skaters have a way to determine the radius of the used blade and then compare their measures with the blade's manufactured radius (as determined by internet research), they can make informed choices about whether the used blades are still suitable for their needs. In the activity described in this article, quantitative ways of measuring skate blades' radii are explored.

In Model-Eliciting Activities (MEAs) designed for instructional use in mathematics classes, students are asked to make mathematical descriptions of meaningful situations (Lesh & Doerr, 2003). The Common Core State Standards identifies being able to model with mathematics as one of the eight Standards for Mathematical Practice (National Governors Association Center for Best Practices, 2010). Many MEAs have been developed for classroom implementation (e.g., Bliss et. al., 2016; Blum et. al., 2007), and the present article adds to this literature base by providing another MEA that can be used.

Within the geometry standards, students are specifically expected to be able to "use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder)." Sporting equipment is a rich context for MEAs, as examining sporting equipment can help students both integrate their existing geometric concepts with their knowledge of the sport and learn new mathematical concepts in an applied situation. For example, Magiera (2013) describes how middle school students solved an MEA about the materials used to construct baseball bats using area estimation. While solving the MEA, Magiera's (2013) students were given three microscopic images of aluminium samples and were asked to determine the typical size of the aluminium crystal in each image, as the size of crystals relates to the strength of material found in an aluminium bat. Since the crystal shapes were irregular polygons and not all crystals were the same size, students developed their own ways to model each aluminium sample's crystal size.

Another geometrically interesting piece of sporting equipment is the figure skate blade. From a side view, a figure skate blade is a portion of a circle's sector (see Figure 1). In Figure 1, arc $\widehat{BC}$ begins at the heel of the blade and ends at the part of the blade that would correspond to the top of the skater's foot arch. The remaining portion of the figure skate blade, from point C to the toe pick of the blade, has a different radius.



1 Figure 1. The figure skate blade has circular curvature.

This blade shape allows figure skaters to perform intricate turns and spins that they would not otherwise be able to complete if the blade were completely flat. If a skater is considering purchasing a used figure skate blade, the skater can determine whether the used blade still has the same radius as a newer version of the skate blade.
The radius of the circle that forms arc $\widehat{BC}$ can be determined using a variety of strategies. The key question in this MEA is, "How can we determine the radius of the circle from which the figure skate blade was cut?" The Common Core State Standard that most closely aligns to this activity is HSG.C.A.2, "Identify and describe relationships among … radii and chords (National Governors Association Center for Best Practices, 2010)." Our research questions are the following:

- What mathematical ideas do participants use to solve the key question?
- What representations do participants use to solve the key question?
- What are benefits and drawbacks of these representations?

In this article, we will explain how solving the problem about the figure skate blade radius is an MEA. We describe our diverse set of participants who worked on this MEA, and describe the different settings in which we implemented the MEA. We present our results – the various strategies that participants used to solve the problem. Finally, we discuss possible extensions to the activity.

Models and Modeling Perspective for this Model-Eliciting Activity

The main reason for including mathematical modeling activities within classroom instruction is to help students learn mathematics in engaging settings (Zbiek & Conner, 2006). In Model-Eliciting Activities, students solve applied problems by making mathematical sense of the problem while developing a sensible solution (Lesh & Doerr, 2003). Students' learning of mathematics can take place through the process of building models (Lesh & Zawojewski, 2007). Sometimes, answering modeling problems involves students' researching the situation for themselves, making some reasonable assumptions and strategies for answering the question; people who look at the same modeling problem may have multiple valid solutions based on the different assumptions that were made (Bliss et al, 2014).

The modeling process involves students' defining the problem to be solved, making reasonable assumptions to help simplify the problem, defining the variables related to the problem, obtaining a trial solution to the problem, analysing the strengths and weaknesses of the model, and reporting the results (Cirillo et al, 2016). Often, students' initial representations are revised when they check whether the solution they obtained was feasible; the modeling process is iterative, since it repeats when students evaluate their model and refine it (Usiskin, 2015).

In the Radius of the Figure Skate Blade MEA, participants have been given the problem taken from the external context, so the first step of the modeling cycle has been completed for them. Yet, participants still need to identify the portion of the figure skate blade for which they want to determine the radius (referring to Figure 1, determining the location of arc $\widehat{\boldsymbol{BC}}$). Figure skate blades are typically constructed using two or more different circles' radii, with one larger radius used for the back part of the blade and then another smaller radius for the portion of the blade closer to the toe pick. Participants can take measurements of the blade, or trace a cross-section of the blade onto a sheet of paper. They can use geometry theorems and relationships between the quantities that they measured to develop a solution to their problem. If two strategies give matching answers, it makes more likely the solution is correct.

Below, a table shows the five steps of the mathematical modeling cycle identified by Usiskin (2015) and described by Usiskin (2015) and Anhalt & Cortez (2015) as they apply to this MEA:

Modeling process steps (Usiskin, 2015)	**What completing the step entails (Usiskin, 2015; Anhalt & Cortez, 2015)**	**What participants in this MEA do**
1. Choose the real problem and simplify it	Identify a problem taken from an everyday life context that must be solved	The problem has been provided for the participants. Based on a given figure skate blade, participants decide the portion of the blade that they will use to solve the problem. They trace this arc.
2. Find a mathematical model for the simplified problem	Determine all given information and what assumptions are necessary. Translate this information into a mathematical problem that can be solved.	Select a strategy that can be used to determine the center of a circle given an arc.
3. Solve the problem within the mathematical model	Solve the mathematical problem, analyze and perform operations in the model.	Using measurements from the arc and auxiliary chords or other line segments that might be drawn, participants apply any theorems that were identified.
4. Translate the solution back into the real-world situation	Interpret and reflect on whether the mathematical answer makes sense in terms of the original situation.	Participants locate the center of the circle and represent the radius (either by a piece of string of the same length, or using a line segment drawn in dynamic geometry software)
5. Check whether the solution is feasible, and if not, go back to step 1 or 2.	Reflect on whether the results make sense. Revise the assumptions made according to what was learned in the first solution. The type of mathematics used in the current model may be different from what was used in a previous model.	Participants check their solutions by seeing whether the radius length in their solution in fact traces the arc of the figure skate blade.

In Appendix A, a student handout is provided for the activity with instructions guiding students through the modeling cycle as described above. Depending on the materials that are available to the participants, there are two version of the activity. One version involves a hands-on approach with ruler, ribbon, and a compass. The second version involves use of dynamic geometry software.

Materials and Methods

The participants in this study were students in five sections of a course entitled "Perspectives on Science and Mathematics," which is a required course for undergraduate students pursuing teaching certification in secondary and middle school mathematics and science. The first author was the instructor for this course, and the second author helped implement the lesson. The sections were taught in Fall 2017, Spring 2018, and Fall 2018. The course is also a university-wide core course in Arts and Humanities, and as such, enrolment in the course is open to non-education majors. There was an average of 15 students in each section. The MEA was implemented in one hour-long class period within a two-week long unit about geometry related to cultural artifacts. Prior to this MEA, various proofs of the Pythagorean Theorem were considered, and the Golden Ratio was discussed as it related to various artwork including paintings by Leonardo da Vinci. The figure skate blade was presented as a cultural artifact since it was a method of human locomotion in Nordic and northern European countries.

Each group of 2-4 participants was given a different figure skate blade. The blades differed in size and manufacturer, so that between groups the focus of the discussion would be on methods used to solve the problem rather than the precise numeric solutions obtained. Students had access to a library of common mathematics textbooks currently used in local public-school classrooms and a classroom set of laptop computers with geometry software. Other materials provided for the activity included straightedge rulers, measuring tape, craft ribbon and craft string, protractors, pencils, and blank sheets of paper. The students were not given a specific strategy to employ.

Results

To answer the research questions posed, we present an analysis of participants' strategies. Participants were allowed to choose their strategies after discussing with their groups. Each strategy will be discussed in the following manner: First, each mathematical idea will be described using theorems or equations commonly found in secondary geometry textbooks. Secondly, the representations that participants used will be described with accompanying work or photographs of participants implementing their ideas. Benefits and drawbacks of the strategies are subsequently discussed.

All of the participants' strategies started with their tracing the edge of the skate blade onto a sheet of paper. In this section, we will refer to this blade tracing as arc $\widehat{\boldsymbol{BC}}$. Figure 2 shows how one student held the blade flat along the side of the table to trace it onto the paper.

Figure 2. Blade tracing onto paper.

Strategy 1: Experimenting Along Perpendicular Bisector

1.1. Mathematical Description

The theoretical bases for this strategy are that the center of the circle lies on the perpendicular bisector of a chord, and that the center of the circle is equidistant from all points on the circle. Referring to Figure 3, the participants using this strategy would first trace the skate blade ($\widehat{BC}$) onto a sheet of paper and then construct $\overline{AE}$, the perpendicular bisector of chord $\overline{BC}$. One participant stays at point B with one end of a string, while a second participant stays at point C**.** A third participant moves along $\overline{AE}$ with the other end of the string (of variable length). For each possible location of the circle's center that the third participant travels along perpendicular bisector BC, the first and second participants experiment to see whether the string traces arc BC exactly. If not, the third participant relocates along BC. Participants attempted to find at which point D along $\overline{AE}$ the string will trace $\widehat{BC}$ exactly (see Figure 4).

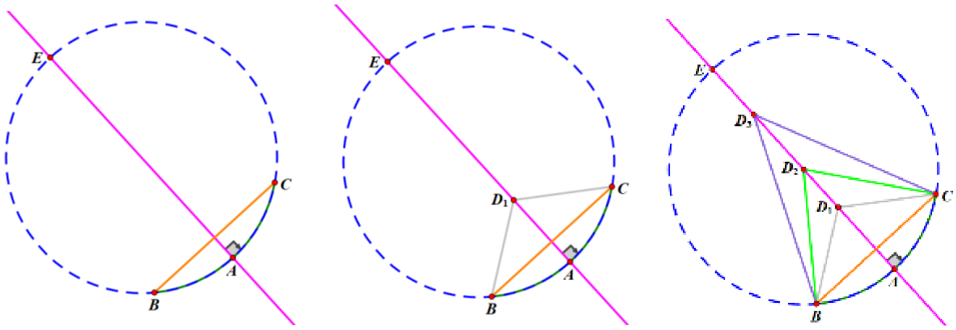


Figure 3. Experimenting along perpendicular bisector.

1.2. Participants' representations

Figure 4. Participants' teamwork using the experimenting along perpendicular bisector strategy.

Strategy 2: Perpendicular Bisectors of Chords

2.1. Mathematical Description

The theoretical bases behind this strategy are that the perpendicular bisector of a chord passes through the center of the circle, and the perpendicular bisectors of chords within the same circle will intersect at the center of this circle. Referring to Figure 5, given arc $\widehat{BC}$ and another point A on arc $\widehat{BC}$, the perpendicular bisectors of chords $\overline{AB}$ and $\overline{AC}$ intersect at point D, which is the center of the circle.

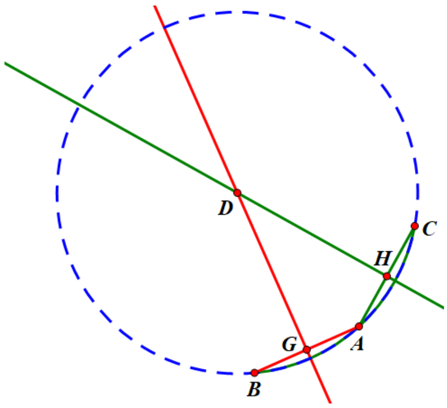


Figure 5. Perpendicular bisectors of chords

2.2. Participants' representations

One group of participants used paper folding and string to construct the perpendicular bisector of chord $\overline{\boldsymbol{AB}}$ through midpoint G, and then they constructed the perpendicular bisector of chord $\overline{\boldsymbol{AC}}$ through midpoint H. They inserted the string into the paper crease that was folded to bisect the two chords $\overline{\boldsymbol{AB}}$ and $\overline{\boldsymbol{AC}}$. They then extended the string until the two strings intersected, and they used tape to affix the string to the table (see Figure 6, left hand side).

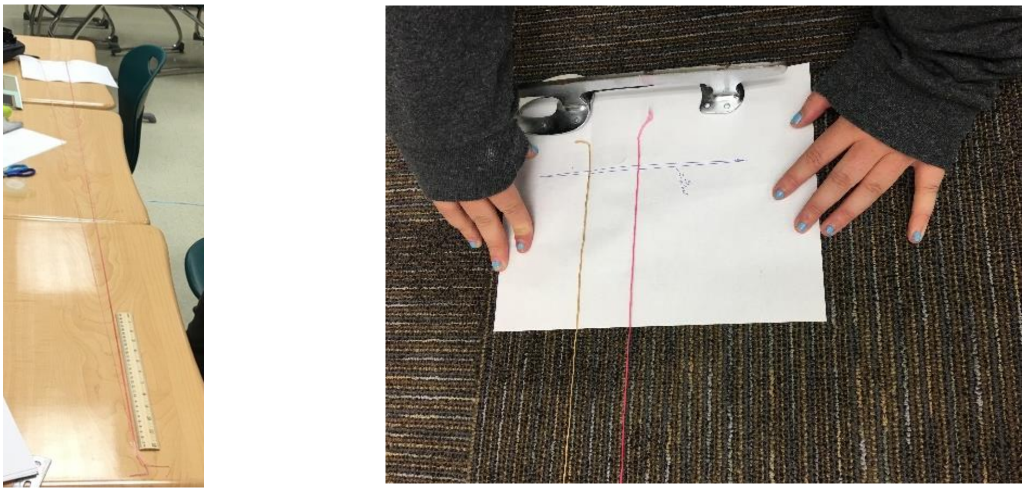


Figure 6. Participants' teamwork using perpendicular bisector of chords strategy.

Another group of participants used the string and paper folding on a different set of chords, $\overline{BC}$ and $\overline{AC}$. Instead of using table tops, they used the floor as a flat surface. (See Figure 6, right hand side).

Strategy 3: Tangent Theorem

3.1. Mathematical Description

The tangent theorem strategy uses the idea that a tangent line $\overrightarrow{\boldsymbol{FA}}$ to $\widehat{\boldsymbol{BC}}$ is perpendicular to the diameter $\overline{\boldsymbol{AE}}$ drawn to point of tangency A.

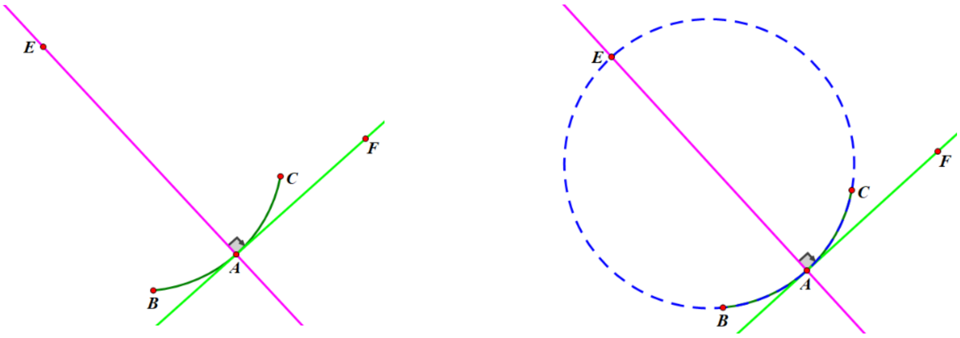


Figure 7. Tangent to arc.

3.2. Participants' representations

When implementing this strategy, participants constructed $\overrightarrow{FA}$ and its perpendicular bisector along $\overline{AE}$ on the sheet of paper that included $\widehat{\boldsymbol{BC}}$. Then they taped this sheet of paper onto the white board on the wall. Participants then traced $\widehat{\boldsymbol{BC}}$ onto a different sheet of paper (see Figure 8, left hand side), which was then used as a stencil with which to extend the arc containing $\widehat{\boldsymbol{BC}}$ along the white board. They recognized that when the perpendicular bisector of $\widehat{\boldsymbol{BC}}$ intersects with the circle a second time, at point E, then $\overline{\boldsymbol{AE}}$ is the diameter of the circle (see Figure 8, right hand side).

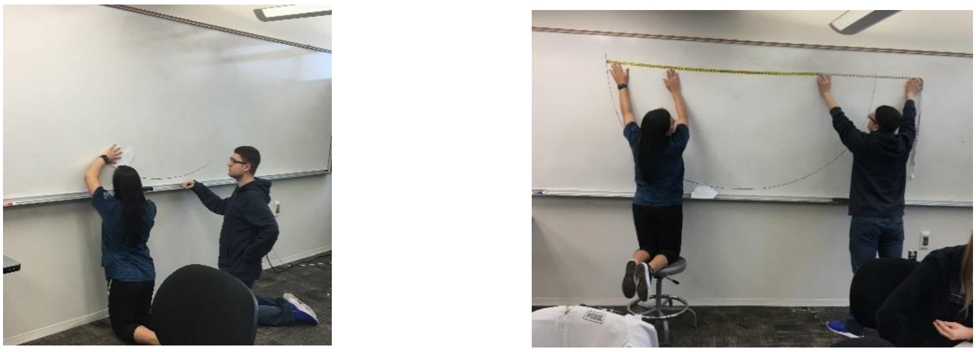


Figure 8. Participants' teamwork using tangent theorem strategy.

Strategy 4: Intersecting Chords

4.1. Mathematical Description

The theoretical basis of this strategy is that for chords $\overline{EA}$ and $\overline{BC}$ which intersect at point G, (see Figure 9), $\overline{EG} \times \overline{AG} = \overline{BG} \times \overline{GC}$; and the perpendicular bisector of a chord coincides with the circle's diameter. Lengths $\overline{BG}$, $\overline{GC}$, and $\overline{AG}$ can be measured with a ruler. Based on the intersecting chords formula, $\overline{EG} = \frac{\overline{BG} \times \overline{GC}}{\overline{AG}}$, and $\frac{\overline{EG} + \overline{AG}}{2}$ is the radius of the circle.

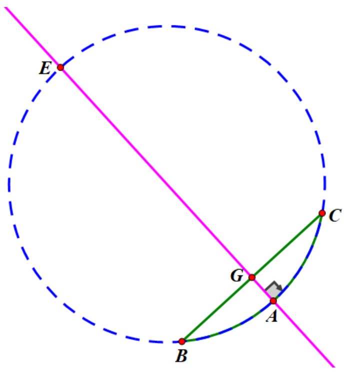


Figure 9. Intersecting chords.

4.2. Participants' representations

Referring to Figure 10, participants first draw chord $\overline{BC}$ connecting the endpoints of their blade tracing $\widehat{BC}$, and find the midpoint G – sometimes this is measured by a ruler, but most frequently this is accomplished by folding the sheet of paper on which $\overline{BC}$ is drawn in half. Segment $\overline{EA}$ is thus the constructed perpendicular bisector of $\overline{BC}$, which is the crease of the sheet of paper that was folded to bisect $\widehat{BC}$.

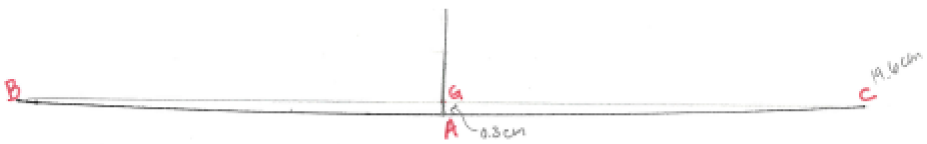


Figure 10. Participant's work using intersecting chords strategy.

In the work shown in Figure 10, $\overline{BC}$ was 19.6 cm, so, $\overline{GC}$ and $\overline{BG}$ were both 9.8 cm each. $\overline{AG}$ was 0.3 cm. The resulting diameter, as determined by the intersecting chords theorem, was 323.43 cm. The radius of the circle was thus 160.215 cm.

Strategy 5: Pythagorean Theorem

5.1. Mathematical Description

After the perpendicular bisector of $\widehat{BC}$ is drawn through point M, the midpoint of $\overline{BC}$, there exists a right triangle whose hypotenuse is the radius of the circle (see Figure 11). The center of circle, point D, lies on the perpendicular bisector of $\overline{BC}$. Using the Pythagorean Theorem on the side lengths of right triangle DMC, $\overline{DM}^2 + \overline{MC}^2 = \overline{DC}^2$. The length $\overline{MA}$ can be measured using a ruler. If the radius r = $\overline{DC}$, and $\overline{DM}$ = r - $\overline{MA}$, a restatement of the Pythagorean Theorem is (r - $\overline{MA})^2 + (\overline{MC})^2$ = r^2.

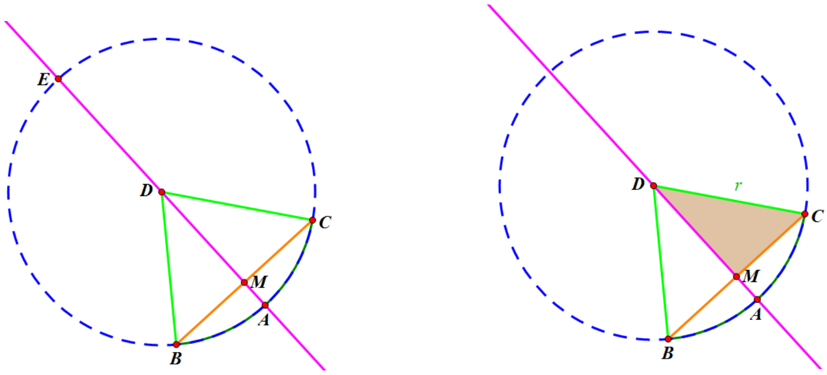


Figure 11. Right triangle DMC.

5.2. Participants' representations

In Figure 12, a participant's handwritten work is shown. The participant measured $\overline{MA}$ = 2/16 inches, and $\overline{MC}$ = 3.5 inches. The participant then solved (r - $\frac{2}{16})^2 + (3.5)^2$=r^2 and found that r = 49.0625 inches.

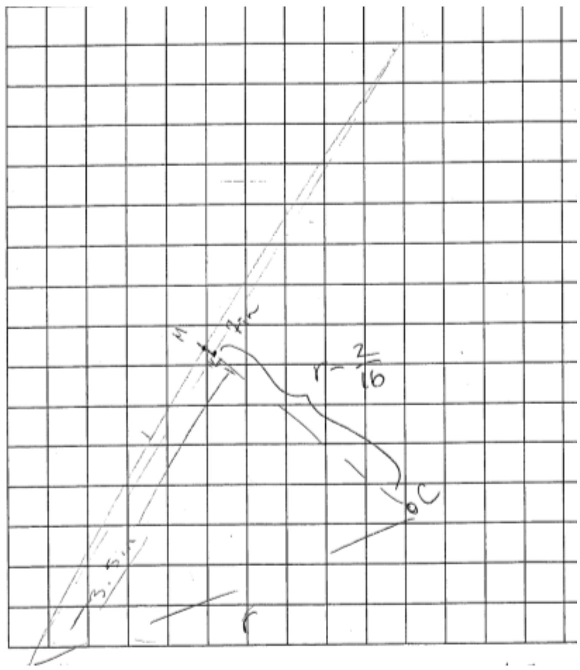


Figure 12. Participant's work using Pythagorean Theorem.

Strategy 6: Arc Length Formula (Unproductive)

Another strategy which participants attempted to use was the arc length formula, S = □□r. The arc length, S, can be found using measuring tape. However, it is not possible to use this formula because there are two unknown quantities, the central angle □ and radius r. Both the central angle and the radius depend on the location of the circle's center, which is also unknown.

Benefits and drawbacks

Strategies 1, 2, and 3

Using strategies 1, 2, 3, participants need to cooperate as a group in order to attain solutions; everyone in the group needs to be actively participating. Also, for the first three strategies, participants need to understand the strategy conceptually prior to recognizing where the solution lies.

The drawbacks of using strategies 1, 2, and 3 are primarily due to physical limitations. Since the radii are typically between 4 to 10 feet, a considerable amount of flat space is needed. Participants may need to move tables and chairs away in order to clear space on a classroom floor, use a long hallway, or connect tables together to use the tabletops as the flat surface. Participants who have limited mobility may have a difficult time fully participating in the activity. Also, the string that is used must be kept taut when attempting to find the location of the circle center, otherwise it no longer represents a straight line.
One benefit of using the first strategy is that when participants have found the correct location of the circle's center, they can verify that one end of the string exactly traces the figure skate blade when the other end of the string is placed at the center of the circle. Since strategy 1 involves much trial and error, participants often wonder what other strategies are being used by the other groups to solve the problem.

Strategies 4 and 5

There are many benefits of using strategies 4 and 5. The measured lengths are smaller than the sheet of paper on which the blade was traced, and these measurements can be easily taken with a standard 12-inch ruler. While it might take participants a long time to consider using these strategies, once they recognize them as ways that will produce a solution, the time involved in implementing these strategies is relatively small. These two strategies are interesting applications of algebraic manipulations that students may be able to appreciate.

A drawback of strategies 4 and 5 is that participants do not need to physically find the center of the circle in order to use this strategy. Another drawback of these strategies is that if participants are given the relevant formulas, participants can obtain a solution without much conceptual understanding by simply substituting measurements into a formula. Also, if students' skills in algebraic manipulation (such as factoring quadratic equations) are not strong, they may have trouble generating a solution.

Discussion

The key question in this MEA relates to real life since participants used real sporting equipment that is used in figure skating. Solving the key question involves mathematical modeling because participants generated a mathematical model of the figure skate blade. In MEAs, typically many different representations are used within the solution of the problem (Lesh and Doerr, 2003). Figure 13 shows examples of several types of representational media that were used by participants in this activity, including equations, diagrams, concrete models used while solving the problem, and experience-based metaphors.

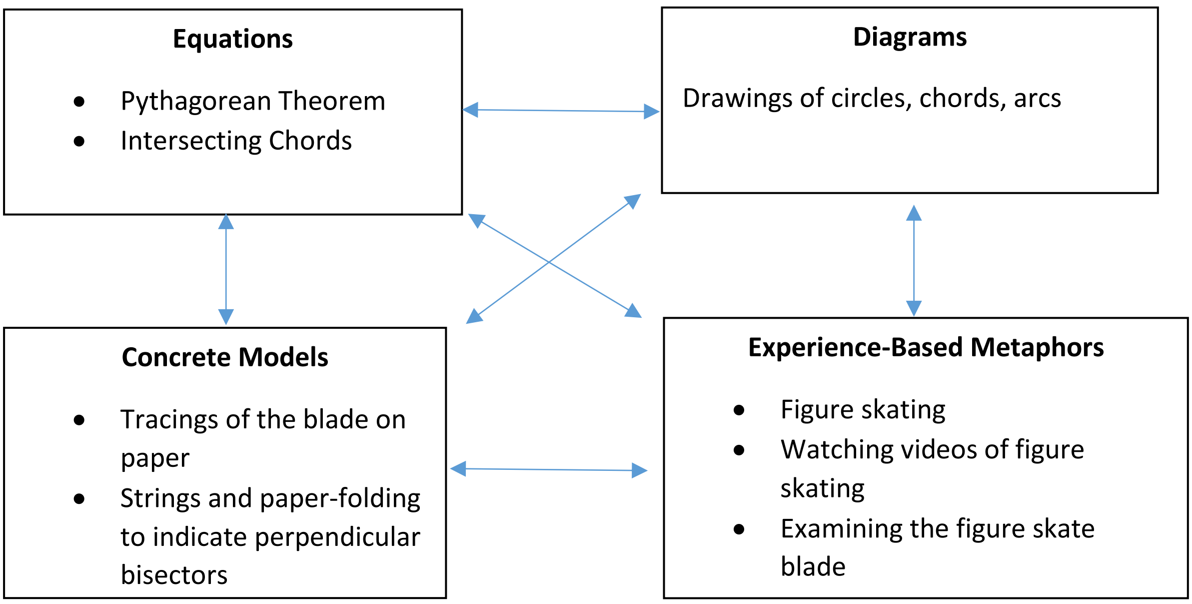


Figure 13. Representational media in this MEA.

In this MEA, participants used various algebraic and geometric ideas, reported as Strategies 1 through 5 in the Results section, in order to find an estimation of the radius of the arc of a given figure skate blade. There are other viable strategies that were not used which were not used by these participants. For example, it is possible to create a triangle with three points identified on the traced arc, and find the incenter of the triangle, which is also the center of the circle.

By going through the modeling process, students must apply geometry content that they have learned to a new context. To further broaden students' perspectives on different solution processes, instructors can facilitate discussions to have students think metacognitively about their solution processes. For example, students could reflect on the effectiveness and limitations of the various solution paths pursued within the class. Chamberlain and Moon (2005) provide a Quality Assurance Guide that could be used to critically evaluate each solution method based on how useful it is and how easy it is for others to adapt it for their situations. Modifications to the MEA can be made based on the population of participants solving this problem. To make the MEA more challenging, instructors can ask students to write and solve their own problems to describe the blades. If the cognitive difficulty of the problem is too high, some strategies and applicable geometry theorems can be provided to the participants. For example, the first author raised this key question with elementary and middle school students attending a summer figure skating camp, and most of these students had not yet learned any of the applicable geometric theorems within their grade school curriculum. So, for these students, it would have been difficult to complete step 2 of the modeling cycle of finding their own mathematical models. Yet, the students were able to determine the size of their own figure skate blades using calculators and rulers when they were given strategy 4, the intersecting chords theorem. These students were able to complete steps 3 through 5 of the modeling cycle after they were given this theorem.

Another modification to this MEA can be made based on method of delivery. This activity involves physically being able to use a skate blade and find a mathematical model to describe it. If MEA needs to be altered for online instruction, several modifications can be made. The instructor could supply photographs of the figure skate blades, and the students could use online geometry software rather than the other materials that were described in this article. Or, students could be provided the option of finding any other circular objects for which they can easily trace an arc, and a comparable analysis can be conducted.

This article demonstrates one MEA in which sports can be used to solve a middle and secondary level mathematics problem. Other MEAs for middle and secondary level mathematics students have been created using sporting equipment as their contexts – for example, the tessellating of shapes that comprise a soccer ball can be studied geometrically (SGMM, 2006). Many more MEAs can similarly be created using sports contexts as a springboard for applying geometric and/or algebraic concepts. Circles and arcs of circles are common shapes present in other sports beyond figure skating. To extend this activity, other sporting contexts at have circular arcs can be examined. For example, other pieces of sporting equipment such as pucks used in the ice hockey, discuses used in track and field, and frisbees used in ultimate frisbee shapes with circular cross-sections. The "cesta" or basket scoop used to play the Spanish handball game of jai alai is comprised of a curved surface and can be analysed geometrically as well. In tetherball, a spherical ball is tied with a string to the top of a vertical pole, and the ball travels in a circular path around the pole.

Beyond sporting equipment, cross sections of surfaces where sports are played can also be circular. For example, the side view of a snowboard half-pipe is an arc. If a snowboarding official wanted to determine whether the half-pipe that was constructed for a competition satisfies regulation sizes, the official could use similar mathematical reasoning to find out whether the specifications for height and radius are satisfied. Race car drivers may wish to find a quantitative way to evaluate the circular curves of their racetracks from a bird's eye view, in order to assist them in their planning of speeds they plan to use while navigating their paths. There exist tracks for motorcycles, bicycles, and skateboards which are based on the shape of full circles or arcs of circles. In surfing, ocean waves sometimes form circular loops and surfers may want to describe the magnitude of their waves by the diameter of the wave. In bowling, officials measure the curvature of bowling alleys to the thousandth of an inch, since flatter surfaces are more suitable for competitive bowling.

Athletes' bodily motions that are circular in nature can also be used in MEAs. Dancers frequently rotate their bodies in circular paths about a vertical axis as they perform turns. Gymnasts rotate their bodies along a horizontal. axis as they execute flips or swing on horizontal bars. Athletes throwing a shot put rotate within a circular ring and travel in a spiralling circular motion prior to releasing the shot put. Golfers use a somewhat circular swinging action with their golf clubs

Declaration of Interest Statement

The authors report no conflict of interest.

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