Mathematics and Sports

A mixed-integer linear programming approach to constructing sporting fixtures with minimal inequity

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Abstract
Ideally, sporting fixtures should be equitable to all teams. This typically means each team plays all other opponents an equal number of times, and other inherent inequities such as a home-field advantage are shared equally across the fixture. However, in some instances this is not possible. Time constraints may prevent teams from playing all other teams an equal number of times, while venue constraints may only permit a subset of the teams to play in each round. Challenges such as these are considered, and a mixed-integer linear programming formulation which takes such issues into account is proposed. Two examples are given which demonstrate that the proposed formulation can be solved to provable optimality in tractable time on modern machines, while heuristic approaches could be used to rapidly obtain high-quality solutions if needed.

Keywords: Sporting, Fixtures, Mixed-integer linear program, Inequity

1 Introduction

When scheduling a fixture for a sporting competition, it is important to ensure that, whenever possible, no individual team (or competitor) is unfairly advantaged or disadvantaged by the fixture, or if such advantages are unavoidable, that they are minimised. For major sporting leagues, significant money is invested by both the league and the various teams, and fair fixtures are essential for maximising revenue and fan engagement in the competition [8]. Even for local or community sports, where player engagement is paramount, it is important that fixtures are viewed as being as equitable as possible, despite there often being tighter constraints imposed by the lack of available funding or venues [4].

A common method for scheduling is to use a round robin fixture, in which each team is scheduled to face every other team once. It is assumed there is an even number n of teams; if not, the standard practice is to introduce an additional, artificial, team which acts as a bye [12, 13]. The fixture is then divided into $n-1$ events, where each team competes once per event (unless they are paired with the bye team). In instances where the venue of play is likely to have an influence on the result (a so-called "home field advantage"), it is common for each team to be scheduled to face every other team twice (or any even number of times), with one (or half) of the contests at each of their preferred venues. Typically, such a fixture is easy to construct from a standard round robin fixture, by simply repeating the latter as many times as desired and alternating the venues. Of course, additional constraints such as venue availability can add to the complexity, but standard integer programming software such as IBM ILOG CPLEX [5] is able to handle these constraints with little difficulty.

However, due to time constraints, it is sometimes not possible to have every team face all other teams an equal number of times during the course of a season. This is particularly an issue during local or community sports, where availability of venues and limited funding can impose various restrictions. However, it is also an issue for major sporting leagues, for which we give a few examples below. In each of the following examples, it is common for many teams to only play each other once, and since teams are typically based in different cities, one of the teams gets to enjoy being the "home team". It is established that this represents a strong advantage [10, 11, 14].

National Football League (NFL): The NFL is the highest professional American football league in the world, and consists of 32 teams split into two conferences. During the regular season, each team plays 17 matches and has one bye. The fixture is uneven since it involves playing three particular teams (from within the same conference) twice, as well as six other matches against teams in the same conference, and five matches against teams in the other conference. Note that this means that many teams do not face each other at all during the regular season. This is particularly important since only the top four teams per conference are guaranteed entrance into the playoffs, along with another three “wild card" teams from each conference. This fixturing disparity is exacerbated by teams experiencing significantly different amounts of rest days, travel distances, and a sizable home-field advantage [9].

Major League Soccer (MLS): The MLS is the highest professional football (soccer) league in the United States. As of 2023, it consists of 29 teams, split into the Eastern conference and the Western conference, containing 15 and 14 teams respectively. During the regular season, each team plays 34 matches, with teams playing each team in their conference twice, and one game against some (but not all) of the teams in the other conference. It is worth noting that the phenomenon of teams not playing every other team at least once is fairly new in MLS; it happened for the first time only in 2020, when the number of teams increased from 24 to 26. The nine highest-ranked teams from each conference then move onto the MLS Cup playoffs.

Australian Football League (AFL): The AFL is the top-level competition which plays Australian Rules Football, a sport which despite being unique (in a commercial sense) to Australia is nonetheless the most popular sport in that country. Indeed, the AFL brand is estimated to be worth more than a billion Australian dollars. The AFL consists of 18 teams, and has a minor round fixture consisting of 24 weeks. Each team plays 23 matches against the other 17 teams, plus one bye. Hence, each team plays six of the teams a second time. Since the teams are not equally skilled, the AFL fixture is inherently inequitable, with some teams required to play a difficult opponent twice, while other teams have the benefit of facing a weak opponent twice. This also means that most teams only play each other once, exacerbating the home-field advantage. Other issues include teams playing one opponent twice before facing a different opponent for the first time. Interested readers can find more about the unevenness of AFL fixtures in [1, 2, 6].

Other issues can arise that make fixturing difficult, such as an insufficient number of venues. Typically, if there are n teams, there will be $\frac{n}{2}$ contests per event, spread among m venues. If $m < \frac{n}{2}$ then it may be necessary to schedule several contests at a single venue which can also add to the time constraints. Alternatively, for some sporting fixtures, such as local sporting competitions, it is standard for all contests within an event to occur simultaneously. In such a case, if the number of venues is insufficient to accommodate all of the teams, then the only solution is to schedule multiple byes per event which, again, can induce additional time constraints to ensure that all teams are able to play the same number of matches over the course of the season.

In the event that time constraints prevent a fixture from being built from a single round robin fixture repeated multiple times, two issues arise. The first is inequity as discussed, and the second is that the fixture may lose some desirable characteristics. We propose some examples of desirable characteristics as follows:

- If a team is to have multiple byes, it is best if they occur at roughly equal intervals. This helps to ensure that each team has roughly the same expectations in terms of the number of matches they must play between rests.
- Teams should face every other team once before they start facing teams for a second time. This helps to prevent the situation where one team is unfairly advantaged by playing a weak team multiple times before playing a strong team (or vice versa).
- At any given stage in the season, no team should have competed in significantly more matches than another team. This helps to preserve the integrity of the ladder at any given stage in the season.
- It is better to have fewer events with more contests per event, than more events with fewer contests (i.e. minimise the number of byes per event). This ensures the teams are able to participate more often within a period of time.

The above discussion leads to the following question: can we design a mixed-integer linear programming formulation which will produce fixtures in which the inequity is minimised, and a chosen set of desirable characteristics is maintained? In what follows, we will introduce such a formulation, and explain how certain aspects of it can be activated or deactivated as desired.

2 Mixed-integer linear programming model for minimising inequity in fixtures

Optimisation problems can often be formulated as *mathematical programs*, which can be loosely understood as the following. Given a set of *decision variables* whose value is to be determined, some *objective function* of those decision variables is to be minimised (or maximised) subject to a set of *constraints* which must all be simultaneously satisfied. If all of the variables are permitted to be continuous, and the objective function and all of the constraints are linear functions, then the mathematical program is said to be a *linear program*, and it can be efficiently solved [7]. However, if some (or all) of the variables are further constrained such that they must take integer values, such a program is called a *mixed-integer linear program* (MILP), and there is no known efficient algorithm which is guaranteed to solve it. Nonetheless, there are highly developed software packages, such as IBM ILOG CPLEX [5] which are often successful in practice for large MILP formulations. For more information about MILP optimization and its many applications, the interested reader is referred to the excellent book on the subject by Floudas [3].

In this section, we will propose an MILP formulation which seeks to produce a fixture in which inequity is minimised. It will be assumed that there are n teams, m venues, and e events. Note that n can be odd or even here. It is assumed that each venue can only be used once per event (if this is not the case, the venue can be thought of as multiple venues, one for each time it is available). Hence, it is necessary for there to be at least $n-2m$ teams scheduled to have a bye in each event. It is also assumed that each team may only receive

Variable set	Indices	Description
x_{ijk} (binary)	$i = 1,\dots e$, $j,k = 1,\dots,n$,	equal to 1 if team j is to face team k during event i. equal to 0 otherwise.
y_{jk} (integer)	$j,k = 1,\dots,n$	the number of times team j is scheduled to face team k.
g_i (integer)	$i = 1,\dots,e$	the number of matches scheduled in event i.
b_{ij} (binary)	$i = 1,\dots,e$ $j = 1,\dots,n$	equal to 1 if team j has a bye in event i. equal to 0 otherwise.
h_{ij} (binary)	$i = 1,\dots,e$ $j = 1,\dots,n$	equal to 1 if team j is the “home" team in event i. equal to 0 otherwise.
c_{ij} (integer)	$i = 1,\dots,e$, $j = 1,\dots,n$	equal to the number of byes team j has had by the conclusion of event i.
m_{ijk} (integer)	$i = 1,\dots,e$ $j,k = 1,\dots,n$	equal to the number of matches played between teams j and k by the conclusion of event i.
z (integer)		the minimum number of matches played by each team.

Table 1: The sets of decision variables which will be used in the MILP formulation described in this section. In each case, the name of the variable set is given, along with a description. Each variable set is denoted as containing either binary variables (restricted to either 0 or 1) or integer variables, and the relevant ranges are given for its indices.

a maximum of one bye during a period of t events, and that each specific pairing of teams may only occur once during a period of w events. Finally, we will use the parameter d_{jk} which is set to 1 if teams j and k share the same home venue, and 0 otherwise.

The upcoming linearly-constrained integer program will be designed to avoid undesirable fixturing choices, such as two teams being scheduled to play each other twice in a short time, or one team having lots of byes before other teams have any. The question of minimising inequity will be handled by a suitably chosen objective function, discussed in the next section.

The MILP formulation developed in this section will use the set of decision variables listed in Table 1.

Depending on the scenario, it may be preferable for z be a parameter rather than a variable if the number of matches to be played is a fixed requirement. Likewise, it might be desirable for m to be a variable, in order to the minimum number of venues that need to be hired.

In what follows, we describe a set of constraints (1)–(21). After, we explain each constraint and indicate which are vital, and which can be ignored if certain fixturing features are not required.

$$
\begin{aligned}
x_{ijj} &= 0, && \forall i = 1,\dots,e; \quad j = 1,\dots,n, && (1)\\
\sum_{j=1}^{n}\sum_{k=1}^{n} x_{ijk} &\leq 2m, && \forall i = 1,\dots,e, && (2)\\
\sum_{k=1}^{n} x_{ijk} &\leq 1, && \forall i = 1,\dots,e; \quad j = 1,\dots,n, && (3)\\
\sum_{k=1}^{n} x_{ijk} &= 1 - b_{ij}, && \forall i = 1,\dots,n; \quad j = 1,\dots,e, && (4)\\
x_{ijk} - x_{ikj} &= 0, && \forall i = 1,\dots,n; \quad j,k = 1,\dots,e, && (5)\\
\sum_{i=1}^{e} x_{ijk} &= y_{jk}, && \forall j,k = 1,\dots,n, && (6)\\
\sum_{k=1}^{n} y_{jk} &\geq z, && \forall j = 1,\dots,n, && (7)\\
y_{jk} &\geq 1, && \forall j,k = 1,\dots,n, && (8)\\
\sum_{i=1}^{e}\sum_{l=1}^{n} (x_{ijl} - x_{ikl}) &= 0, && \forall j,k = 1,\dots,n, && (9)\\
\sum_{l=1}^{i} b_{lj} &= c_{ij}, && \forall i = 1,\dots,e; \quad j = 1,\dots,n, && (10)\\
c_{ij} - c_{ik} &\leq 1, && \forall i = 1,\dots,e; \quad j,k = 1,\dots,n, && (11)\\
c_{i+t,j} - c_{ij} &\leq 1, && \forall i = 1,\dots,e-t; \quad j = 1,\dots,n, && (12)\\
\sum_{l=1}^{i} x_{ljk} &= m_{ijk}, && \forall i = 1,\dots,e; \quad j,k = 1,\dots,n, && (13)\\
m_{ijk} - m_{ijl} &\leq 1, && \forall i = 1,\dots,e; \quad j,k,l = 1,\dots,n; j \neq k,l, && (14)\\
m_{i+w,jk} - m_{ijk} &\leq 1 && \forall i = 1,\dots,e-w; \quad j,k = 1,\dots,n, && (15)\\
\sum_{j=1}^{n}\sum_{k=j+1}^{n} x_{ijk} &= g_i, && \forall i = 1,\dots,e, && (16)\\
g_{i+1} - g_i &\leq 0, && \forall i = 1,\dots,e-1, && (17)\\
x_{ijk} + h_{ij} + h_{ik} &\leq 2, && \forall i = 1,\dots,e; \quad j,k = 1,\dots,n, && (18)\\
-x_{ijk} + h_{ij} + h_{ik} &\geq 0, && \forall i = 1,\dots,e; \quad j,k = 1,\dots,n, && (19)\\
\sum_{i=1}^{e} (h_{ij} - h_{ik}) &\leq 1, && \forall j,k = 1,\dots,n. && (20)\\
h_{ij} + h_{ik} + d_{jk} &\leq 2, && \forall i = 1,\dots,e; \quad j,k = 1,\dots,n. && (21)
\end{aligned}
$$

Constraints (1)–(9) are vital constraints which are required to produce a valid fixture. Constraints (1) ensure no team is scheduled to face itself. Constraints (2) ensure that there are not more matches scheduled than there are venues to host them. Constraints (3) ensure that each team is only scheduled at most one match per event. Constraints (4) schedule a bye for any teams without a match during an event. Constraints (5) ensure symmetry of variables (i.e. if team A is scheduled to face team B, then team B should also be scheduled to face team A). Constraints (6) count the number of times each pairing of teams is to face each other. Constraints (7) ensure each team is scheduled to play at least the minimum number of matches. Constraints (8) ensure each team plays every other team at least once. Constraints (9) ensure that each team plays the same number of matches overall.

The remaining constraints may be omitted if desired, as they are only used to ensure desirable features of the fixture. If constraints are omitted, any variables used only in those constraints can be omitted from the model as well.

Constraints (10)–(12) are used to schedule byes in an intelligent way. In particular, constraints (10) are used to count the number of byes each team is scheduled to have had by the conclusion of a given event. Constraints (11) ensure there is no point in the fixture where a team has had two more byes than another team. Constraints (12) ensure that at least t weeks pass after a bye before that team is scheduled to have another bye.

Constraints (13)–(15) are used to prevent certain pairings from reoccurring too often or too early. In particular, constraints (13) are used to count the number of times a pair of teams has been scheduled to have a match by the conclusion of a given event. Constraints (14) ensure there is no point in the fixture where a team has played one team two more times than another team. Constraints (15) ensure that at least w weeks pass after a pairing before that pairing is scheduled again.

Constraints (16)–(17) are used to ensure any events with less than m matches occur at the end of the fixture.

Constraints (18)–(21) are only necessary if there is a need to consider which teams are “home" and which are “away". Constraints (18)–(19) ensure that precisely one team per pairing is considered to be the home team. Constraints (20) ensure that each team be allocated an equal number of home games if possible, or at most one different if necessary (i.e. if each team plays an odd number of matches). Constraints (21) ensure that two teams sharing the same home venue are never both scheduled as the home team during the same event.

Any additional specific requirements can be added to the model as well by adding or modifying constraints accordingly. Some examples of common fixturing difficulties and the constraint to handle them are as follows:

- Only $v < \lfloor \frac{n}{2} \rfloor$ venues are available during week a.
  Solution: Delete constraint (17) for the case when $i = a$, and add the new constraint $g_a = v$.
- A certain team pairing (say, teams r and s) must be scheduled during week a.
  Solution: Add constraint $x_{ars} = 1$.
- Certain teams (say, teams r and s) must play each other a fixed number of times, say f.
  Solution: Add constraint $y_{rs} = f$.
- The home venue(s) for teams $r_1, r_2, \ldots, r_p$ are unavailable during week a.
  Solution: Add constraints $h_{ar_1} = 0$, $h_{ar_2} = 0$, …, $h_{ar_p} = 0$.

3 Minimising Inequity

In order to minimise inequity arising from teams playing each other an uneven number of times, there must first be a measure of quality for each team. Depending on the scenario, this measure of quality could be fairly simple, or quite involved. Several cases are outlined below, with descriptions of how to augment the model (1)–(21) (or any chosen subset of (1)–(21)) appropriately.

Case 1 - Each team j has a quality rating q_j that is independent of any other factors. Then the total quality that team j is scheduled to face over the course of the fixture can be represented linearly by the expression $\sum_{k=1}^{n} q_k y_{jk}$. Since each team will play an equal number of matches, the total amount of quality in the fixture is a constant. Hence, a natural method for minimising inequity is to maximise the minimum quality faced over all teams. This can be achieved by introducing a dummy variable, say γ, and adding new constraints:

$$\sum_{k=1}^{n} q_k y_{jk} \quad \geq \quad \gamma, \qquad \forall j = 1, \ldots, n. \tag{22}$$

Then, if the following optimisation problem is solved,

$$\max \gamma, \qquad \text{s.t.} \qquad \text{Constraints (1)–(22)},$$

the optimal solution corresponds precisely to the case when the minimum cumulative quality faced by any team is maximised.

Case 2 - Each team j has a quality rating q_j when playing at home and p_j when playing away. In this situation, we introduce new binary variable α_{ijk} that will be equal to 1 when team j plays at their home venue against team k during event i, and 0 otherwise. Although the definition of α_{ijk} implies the nonconvex relationship $\alpha_{ijk} = x_{ijk} h_{ij}$, this can be modelled linearly by demanding that α_{ijk} be binary and imposing the following linear constraints:

$$2\alpha_{ijk} - x_{ijk} - h_{ij} \quad \leq \quad 0, \qquad \forall i = 1, \ldots, e; \quad j,k = 1, \ldots, n,, \tag{23}$$

$$\alpha_{ijk} - x_{ijk} - h_{ij} \quad \geq \quad -1, \qquad \forall i = 1, \ldots, e; \quad j,k = 1, \ldots, n. \tag{24}$$

Then, similarly to in Case 1, we can introduce a dummy variable γ and add new constraints.

$$\sum_{i=1}^{e} \sum_{k=1}^{n} \left(p_k \alpha_{ijk} + q_k \alpha_{ikj} \right) \quad \geq \quad \gamma, \qquad \forall j = 1, \ldots, n. \tag{25}$$

Then the optimal solution to the following optimisation problem corresponds to the case when the minimum quality faced by any team is maximised:

$$\max \gamma, \qquad \text{s.t.} \qquad \text{Constraints (1)–(21) and (23)–(25).}$$

Case 3 - Rather than a quality rating, q_{jk} is a predictor whereby $q_{jk} = 1$ implies that team j is expected to defeat team k, and $q_{jk} = 0$ implies that team k is expected to defeat team j. Then the aim is to minimise the gap between the minimum and the maximum expected wins over all teams. This can be achieved by introducing a dummy variable γ, and adding new constraints:

$$\sum_{k=1}^{n} \left(q_{j_1k}y_{j_1k} - q_{j_2k}y_{j_2k}\right) \quad \leq \quad \gamma, \qquad \forall j_1, j_2 = 1, \ldots, n. \tag{26}$$

Then, if the following optimisation problem is solved,

$$\max \gamma, \qquad \text{s.t.} \qquad \text{Constraints (1)–(21) and (26)},$$

the optimal solution corresponds to the case when the gap between the most expected wins and the least expected wins is minimised.

Depending on the situation, this may not be ideal. For example, if there is one team that is expected to defeat every other team and one team that is expected to lose to every other team, the above optimisation formulation provides no value. Neither would any value be obtained from maximising the minimum number of wins. A worthwhile alternative in this situation would be to define $\beta_j := \sum_k q_{jk}y_{jk}$ to be the number of wins obtained by team j throughout in the fixture, and then minimise the sum of squares of β_j. Although this objective function is nonlinear, it is convex and all constraints are linear, so it is of equivalent complexity to solve. The benefit is that the sum of squares objective places a high cost on any one team having a large number of wins, so the desirable outcome corresponds to the case where each team having as similar a number of wins as possible.

Case 4 - Similar to Case 3, q_{jk} is a predictor such that $q_{jk} = 1$ if team j is expected to beat team k when playing at home. Similarly, we also introduce p_{jk} as a predictor such that $p_{jk} = 1$ if team j is expected to beat team k when playing away. By introducing variables α_{ijk} from Case 2 and including constraints (23)–(24), we can then introduce dummy variable γ and add new constraints.

$$\sum_{i=1}^{e}\sum_{m=1}^{n} \left(q_{j_1m}\alpha_{ij_1m} + p_{j_1m}\alpha_{imj_1} - q_{j_2m}\alpha_{ij_2m} - p_{j_2m}\alpha_{imj_2}\right) \quad \leq \quad \gamma, \forall j_1, j_2 = 1, \ldots, n. \tag{27}$$

Then, similarly to in Case 3, the optimal solution to the following optimisation problem corresponds to the case where the gap in expected wins is minimised:

$$\max \gamma, \qquad \text{s.t.} \qquad \text{Constraints (1)–(21), (23)–(24), and (27).}$$

Much like in Case 3, one can define β_j to be equal to the LHS of constraint (27), and then minimise the sum of squares of β_j for a more complex measure.

Case 5 - Similar to Case 3, except rather than a discrete 0/1 predictor, q_{jk} is the probability that team j will defeat team k. An analogous modification is also possible for Case 4. For both of these cases, the models proposed in Cases 3 and 4 are still valid.

4 Example

This manuscript is now concluded with an example of such an uneven fixture. One of the primary features of the proposed formulation is its ability to construct fixtures for situations where the capacity of venues is insufficient to accommodate all teams simultaneously; this is particularly an issue in competitions where the matches must occur roughly simultaneously. For example, consider a local squash competition which takes place at a single sports centre. The centre has four squash courts available, so only four matches can take place per round, but the competition contains ten teams. It is therefore necessary to have at least two byes per week. Note that in this example, it is unnecessary to consider home and away venues, since the competition takes place in a single centre, and so we accordingly switch off that feature in the formulation by removing constraints (18)–(21).

After an independent analysis of each team, their quality is determined to be $q = [50, 45, 48, 43, 40, 43, 47, 42, 49, 40]$. It is decided that each team should have a minimum of five weeks between byes, and a minimum of seven weeks should pass between any pairing being repeated. Since the competition runs twice a year, and the centre is closed over the December/January holiday period, it is determined that each instance of the competition can run for at most 22 weeks, but the competition organisers would like at least 18 weeks to ensure the centre is regularly used. Ideally, the competition organisers would like all four courts to be used each week, or as many as possible if this cannot be achieved. Given that each team is to play an equal number of matches, it is clear that an 18 week competition would only permit each team to play 14 matches, as any more than this requires the use of more than 18*4 = 72 courts. Similarly, team can play at most 15 matches for a 19 week competition, 16 matches for a 20 or 21 week competition, or 17 matches for a 22 week competition. Out of these choices, a 20 week competition is selected as, potentially, it allows a schedule in which all four courts are in use each round, since there are 80 matches to be scheduled and 80 available court slots.

Since there is no need to consider which team is the home team, the model solved is:

$$\max \gamma, \qquad \text{s.t.} \qquad \text{Constraints (1)–(17), (22),}$$

with $e = 20$, $n = 10$, $t = 5$, $w = 7$ and $z = 16$ set as parameters. The model was solved using IBM ILOG CPLEX Optimization Studio v22.1.1.0 on an Intel(R) Core(TM) i7-1270P CPU @ 2.20GHz with 32GB RAM, running Windows 10 Enterprise 22H2 64 bit. The optimal solution was returned in just under five minutes and is as follows:

Round:	**Rd 1**	**Rd 2**	**Rd 3**	**Rd 4**	**Rd 5**	**Rd 6**	**Rd 7**	**Rd 8**	**Rd 9**	**Rd 10**
Court 1:	1 v 2	1 v 8	1 v 10	2 v 5	1 v 5	1 v 6	1 v 7	1 v 4	2 v 8	1 v 3
Court 2:	3 v 9	2 v 4	2 v 7	3 v 6	3 v 10	2 v 10	2 v 9	2 v 3	3 v 7	5 v 8
Court 3:	4 v 5	5 v 7	3 v 8	4 v 7	6 v 8	3 v 4	4 v 8	7 v 10	4 v 6	6 v 7
Court 4:	6 v 10	6 v 9	4 v 9	8 v 10	7 v 9	5 v 9	5 v 6	8 v 9	5 v 10	9 v 10
Bye:	7,8	3,10	5,6	1,9	2,4	7,8	3,10	5,6	1,9	2,4
Round:	**Rd 11**	**Rd 12**	**Rd 13**	**Rd 14**	**Rd 15**	**Rd 16**	**Rd 17**	**Rd 18**	**Rd 19**	**Rd 20**
Court 1:	1 v 9	1 v 2	1 v 8	2 v 10	1 v 7	1 v 4	1 v 6	1 v 9	2 v 3	1 v 3
Court 2:	2 v 6	4 v 5	2 v 7	3 v 8	3 v 6	2 v 5	2 v 9	2 v 8	4 v 6	5 v 10
Court 3:	3 v 5	6 v 9	3 v 10	4 v 7	5 v 9	3 v 9	4 v 8	3 v 7	5 v 8	6 v 8
Court 4:	4 v 10	7 v 8	4 v 9	5 v 6	8 v 10	6 v 10	5 v 7	4 v 10	7 v 10	7 v 9
Bye:	7,8	3,10	5,6	1,9	2,4	7,8	3,10	5,6	1,9	2,4

This solution corresponded to the case where the minimum total quality faced was 714 (by teams 8 and 9) and the maximum was 718 (by team 2). Note that the solution obtained does indeed permit each team to play sixteen matches, and that all four courts are used in each week.

If the same example is run assuming only 19 weeks are used, and therefore each team plays 15 matches, and all other parameters are kept the same, the model is found to be infeasible in 7 seconds, so it is not possible to construct such a fixture that satisfies all the constraints. To handle this, the constraints were relaxed so that teams could be scheduled a bye every four weeks, and pairings could be repeated every six weeks, and the following optimal solution was obtained in just under 69 minutes:

Round:	**Rd 1**	**Rd 2**	**Rd 3**	**Rd 4**	**Rd 5**	**Rd 6**	**Rd 7**	**Rd 8**	**Rd 9**	**Rd 10**
Court 1:	1 v 2	1 v 3	1 v 6	2 v 3	1 v 5	1 v 7	1 v 4	1 v 8	2 v 6	1 v 9
Court 2:	3 v 6	2 v 9	3 v 9	5 v 10	2 v 7	2 v 10	3 v 7	2 v 4	4 v 5	2 v 5
Court 3:	4 v 10	4 v 8	4 v 7	6 v 7	4 v 9	3 v 4	5 v 9	3 v 10	7 v 8	3 v 8
Court 4:	5 v 7	6 v 10	5 v 8	8 v 9	8 v 10	6 v 9	6 v 8	5 v 6	9 v 10	7 v 10
Bye:	8,9	5,7	2,10	1,4	3,6	5,8	2,10	7,9	1,3	4,6
Round:	**Rd 11**	**Rd 12**	**Rd 13**	**Rd 14**	**Rd 15**	**Rd 16**	**Rd 17**	**Rd 18**	**Rd 19**	
Court 1:	1 v 10	1 v 5	1 v 3	1 v 7	2 v 10	1 v 9	1 v 2	1 v 6	2 v 8	
Court 2:	3 v 5	2 v 8	2 v 9	2 v 4	3 v 9	2 v 6	3 v 10	3 v 4	5 v 10	
Court 3:	4 v 6	3 v 6	4 v 8	5 v 9	5 v 6	3 v 7	4 v 6	5 v 8	7 v 9	
Court 4:	7 v 9	4 v 7	6 v 10	8 v 10	7 v 8	4 v 5	8 v 9	7 v 10	—	
Bye:	2,8	9,10	5,7	3,6	1,4	8,10	5,7	2,9	1,3,4,6	

In this case, the minimum total quality faced was 669 (by teams 1, 2, 4 and 8) and the maximum was 674 (by team 5). Note that this fixture, while satisfying all of the desired constraints, is by several measures less desirable than the previous one. For example, the uniform five week gap between subsequent byes for each team in the previous schedule is not achieved here, potentially leading to inequities. Consider that team 6 in the 19-week schedule never plays more than four matches between byes, while team 2 plays a stretch of six consecutive matches from round 12 to 17. Nearly half of the teams get a bye in the final round, which may be seen as an unfair advantage heading into a subsequent finals series.

It should also be noted that in both tested cases, the optimal solution was found within two minutes, and the remainder of the time was spent confirming the optimality. If a good solution is desired and optimality is not crucial, a simple heuristic which continues solving until no improvement is made for a chosen period of time and then returns the best known solution to that point would be able to handle larger competitions with relative ease.

5 Conclusions and Future Work

We have introduced a mixed-integer linear programming formulation which produces fixtures for which inequity is minimised, while also allowing some control over whether certain desirable characteristics of the fixture are preserved. A significant feature of the formulation is its ability to handle fixtures where the number of venues is insufficient to handle all the teams simultaneously, making it particularly useful for sporting competitions with less financial resources available, particularly local or community sports.

In the present model, inequity is modelled solely by providing a quality score for each team, and measuring the sum of quality of all teams being played by each opponent. Minimising inequity is then achieved by maximising the minimum quality faced by any team. It would be a fascinating topic of future research to consider other measures of inequity (for example, taking into account travel distance, or the number of consecutive away matches). One could also consider other ways of minimising inequity, by trying to minimise the gap between the highest and lowest quality faced by each team; such a measure may not lend itself to a mixed-integer linear programming formulation, however.

Although CPLEX was successful at finding optimal solutions for a ten-team competition, it would become intractable if the competition became sufficiently large. In such a case, it would also be fascinating to look into heuristics which attempt to find good solutions to our proposed formulation in an efficient manner, if even such solutions are not optimal.

6 Statements and Declarations

The author has no relevant financial or non-financial interests to disclose.

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