North Carolina Journal of
Mathematics and Statistics
Volume 10, Pages 1–14 (Accepted January 3, 2024, published January 15, 2024)
ISSN 2380-7539

Mathematically Forecasting Stock Prices with Geometric Brownian Motion

ANAY KUMAR, ISA JAMADAR, RADHIKA GOEL, RAM CHARAN PETLURI, AND WEI FENG

ABSTRACT. Predicting the progression of an unsteady stock market appears to be an impossible task due to the volatile nature of investment portfolios. However, principles such as Geometric Brownian Motion account for random occurrences in a way that can be translated to modeling the stock market. This paper analyzes the Reddy-Clinton equation, a difference equation derived by Krishna Reddy and Vaughan Clinton, with the primary intention of modeling stock price movement over time by utilizing existing metrics. The Reddy-Clinton equation incorporates both a certain and uncertain component to generate a figure which effectively depicts the volatility of the stock market. However, this paper aims to clarify the extent of the unpredictability being accounted for by specifically adjusting ϵ, the variable representing stochasticity, through an adjusted bell-curve model. Additionally, the model is calculated over multiple iterations, with the resulting values collectively averaged to increase accuracy. The adapted model was applied to the following five stocks of varying sectors: AAPL, OXY, PYPL, MCD, and SPG, and resulted in a MAPE of merely 6.87% over a 6-month period. Overall, the paper proposes an altered rendition of the Reddy-Clinton equation to better account for volatility and output an accurate model of a stock's performance over a period of time.

Keywords: Stocks, Geometric Brownian Motion, Reddy-Clinton Equation, Unexpected Volatility, Expected Volatility, Capital Asset Pricing Model, Stochastic Modeling

1. Introduction

Investment is an enduring methodology of pursuing financial affluence. A prevalent format of investment is the placement of monetary resources into the stock market, a culmination of publicly traded corporations which private investors can purchase a portion of. These individuals inherently desire to allocate a quantified budget into the most optimal portfolio available to them.

1.1. Stock Market

Given the stock market's inception in 1611, this desire for financial success is a historical constant with investors pursuing maximization of profit through the deliberation of various stock purchases. However, due to the unpredictable nature of the stock market, investors' shares remain in the balance of an unforeseeable future. This paper focuses on the topic of stock pricing, and how a share price changes in relation to time.

Many companies offer the opportunity to own a portion of their company in exchange for a monetary sum. In formal nomenclature, this is dubbed as purchasing a stock for a share price. Companies that offer this sale are defined as publicly traded, as anyone is capable of investing in the company by providing financial compensation to own stock shares. Based on the performance

Received by the editors December 7, 2023.

of a company, shareholders may choose to buy more shares from other shareholders or sell their stake to other investors. As opposed to the provision of a definite price for the transaction, the jurisdiction of the buyer and seller determines a mutually agreeable value for the share price.

This method causes considerable amounts of variation in stock price, hence the generally agreed share price is defined as the market-cap total value of the company divided by the number of outstanding shares. Due to the constant fluctuation of the share price, many investors recognize the potential for monetary gain and employ countless strategies to do so. Despite the numerous strategies that exist, all of these methods strive to predict a stock's future price, an objective that would allow for positive financial yield.

1.2. Indicators

The stock market is rationalized by variables that are capable of indicating a stock's success through the consideration of past performance. A notable facet in the evaluation of a stock's movement is volatility, characterized by how frequently drastic alterations in the unit's price occur. If constant positive and negative oscillations of the closing values are present, the stock is considered highly volatile. The primary metrics used to analyze volatility in this paper are the expected volatility and β values of stock. Volatility is an effective metric in order to define the predictability of a stock's price. Expected volatility utilizes standard deviation of a unit over a period of time in order to assume a certain level of variance, while the β value of a stock evaluates its volatility in relation to the entirety of the market. A β value of 1 is considered to conform with the entirety of the market, while those greater are comparatively volatile and those lesser are relatively more consistent.

Additionally, expected return is a vital component that must be addressed when calculating returns for investments. The expected return largely disregards stock volatility, generating a percentage return that is based on risk-free rates, market conditions, and the share's β value. The formula used for acquiring expected return is dubbed the Capital Asset Pricing Model (CAPM) (Siegrist, 2022). Both expected return and volatility encapsulate the two components of the derived equation, providing certain and uncertain facets, respectively. Logically, this is a cohesive formula, as there is some level of predictability within the stock market, but stochastic variance must also be accounted for.

Fortunately, the econophysics concept of Geometric Brownian Motion clarifies the randomness of stock prices and accounts for arbitrary fluctuations in a more accurate manner. Geometric Brownian Motion (GBM) has been occasionally called "the standard model of finance", and serves as a model to forecast the price of a stock over time (Ibe, 2013). Originally, GBM was adapted from Brownian Motion—a model that references the random motion of particles suspended in a medium—and was implemented into forecasting stock prices, known to be stochastic, or random, by nature (Siegrist, 2022).

Recently, two scientists from the University of Waikato, New Zealand, Krishna Reddy and Vaughan Clinton, expanded on GBM and proposed the Reddy-Clinton equation which serves as the basis of this paper. The equation's primary goal is to model stock price's movement over time. The Reddy-Clinton equation accounts for fluctuations among stocks and helps investors differentiate between times for buying and selling shares (Reddy and Clinton, 2016). However, the Reddy-Clinton equation's use of the vital variable ϵ allows for unpredictability in producing values. Thus, this paper re-calculates ϵ to produce a more accurate stock price prediction model by adjusting a normal bell curve.

2. Definitions

Prior to considering the formatting of the simulation utilized to unveil the performance of stocks, it is vital to develop an initial grasp upon the concepts necessary for comprehension of this process.

2.1. Capital Asset Pricing Model (CAPM)

The Capital Asset Pricing Model (CAPM) is an effective metric in proposing calculations for the expected return value of an investment. Evaluation of this metric is executed using a risk-free rate of return, β values, and general market conditions. The resulting value is used to present a prediction regarding the expectation of a stock's performance through these relatively certain factors. The CAPM is used to determine an idealized version of how an investment will perform, disregarding most factors of variance and indeterminate nature. The CAPM is defined as

$$\mu = r_f + \beta(r_m - r_f) \tag{2.1}$$

where μ is the expected return, r_f is the risk-free interest rate, β is the Beta value or the volatility of the company as compared to the rest of the market, and r_m is the expected return of the market in its entirety, providing insight to the general conditions of the economy. The combination of these factors in this equation defines the expected return value of a stock.

2.2. Geometric Brownian Motion

Brownian Motion is a physics theorem that defines erratic particle movement in a fluid resulting from atomic-level collisions (Feynman, 2013). This principle was translated to economics and titled Geometric Brownian Motion (GBM), a now widely used financial resource in evaluating stock fluctuations. GBM is an effective governing principle in the realm of share prices as stocks are subject to unpredictable variation which requires stochastic modeling. Also, GBM assumes that fluctuations occur randomly in a normally distributed manner, depending on standard deviation (Ibe, 2013). The governing principle in the Reddy-Clinton Equation is GBM, as it allows for the consideration of unexpected volatility (Reddy and Clinton, 2016).

2.3. Reddy-Clinton Equation

The Reddy-Clinton equation is a difference equation that effectively implements components of GBM and CAPM to demonstrate the return of a stock over time. The solution of this equation depends on a certain and uncertain component (Reddy and Clinton, 2016). Although the Reddy-Clinton equation considers how a stock's price can fluctuate—depending on time, the expected return, and the expected/unexpected volatilities—over time, it struggles to accurately forecast stock prices due to the calculation process of obtaining ϵ, the value which presents unexpected volatility. The equation is defined as

$$S_{t+\Delta t} = S_t e^{(\mu - \frac{\sigma^2}{2})\Delta t + (\sigma\epsilon\sqrt{\Delta t})} \tag{2.2}$$

where S_t is the stock price, μ is the value given by the CAPM, Δt is the change in time, σ is the expected volatility which is calculated from standard deviation and an annualized constant, and ϵ is a randomized figure calculated from a normal distribution curve. In the Reddy-Clinton equation, ϵ is calculated from a standard normal distribution curve.

In the Reddy-Clinton equation, the certain component, given as $\mu - \frac{\sigma^2}{2}$, represents a stock's rate of return over a period of time. The uncertain component, defined as $\sigma\epsilon$, represents the random fluctuations in a stock's return by considering the volatility of stock prices. Thus, when these two components are added together, a stock's return is proportionally evaluated in a balanced manner. This is a respectable presentation of stock evaluation, as both variance and consistency are accounted for.

3. Drawbacks

Although the Reddy-Clinton equation suggests an accurate framework for the prediction of stock performance, certain flaws to its procedural method of evaluating economic indicators exist.

3.1. Epsilon (ϵ)

In the Reddy-Clinton equation, the value of ϵ, the representative variable of unexpected volatility, is calculated randomly from a normal distribution curve (Reddy and Clinton, 2016).

Although this procedure has the ability to productively display the stochastic nature of stocks, the Reddy-Clinton equation assumes that various stocks of differing volatility will react with the same tendency for unpredictability. The random calculation of ϵ is a necessity for the effective function of the equation, however, treating all stocks identically in that manner is a method which can be refined.

3.2. Insufficient Iterations

In the original paper, each model prediction only operated with a single iteration of the equation. Effectively, this allows for the presence of extraneous results, as there is no limit to the generated ϵ value, only a strong likelihood of the production of values closer to the mean. However, this does not eliminate the presence of outliers, and one incidental instance of a direly unrealistic ϵ value produces a model with questionable merit.

Although it is vital to allow for extremity in volatility, this process must be executed within reason, as investors may empty their pockets basing their purchasing decisions on a model with greatly varying results. The veracity of a model is greatly lessened if differing results are produced when reiterating the equation.

4. Modifications

To obtain a renovated rendition of the Reddy-Clinton equation, it is quintessential to provide it with certain modifications. These modifications consist of both providing new equations as well as adjusting those which were previously utilized.

4.1. Introduced Equations

The newly conceived equation contains quantities which must be redefined by newly established equations. The primary innovation upon the Reddy-Clinton is the adjustment of the procedure used to obtain unexpected volatility, defined as ϵ.

4.1.1. Selection of ϵ

The impracticality of a standard normal distribution curve for all stocks is resolved by adjusting the ϵ value. This is executed through a developed equation for the evaluation of the standard deviation used to acquire ϵ. This equation considers the expected volatility of a stock when defining the normal distribution bounds which provide ϵ, the unexpected volatility variable. The new standard deviation is calculated by subtracting 0.25 from the share's annualized, or expected, volatility and adding 1 to the resulting figure through the equation

$$a = (\sigma - 0.25) + 1 \tag{4.1}$$

with a defined as standard deviation and σ representing expected volatility. This equation is logically structured, as the average expected volatility of a stock is 0.25, therefore allowing shares with greater volatility to possess deviation values greater than 1, and those with lesser volatility to function with lesser deviation. Effectively, this equation allows for the existence of a direct correlation between a stock's volatility and the utilized standard deviation.

Given this value of a, substitution as a constant into the normal probability density function results in the equation

$$y = \frac{1}{a\sqrt{2\pi}} e^{-1/2(\frac{x}{a})^2} \tag{4.2}$$

which provides a finalized bell curve on the x-y plane. Thereafter, MATLAB is utilized to randomly determine a value from the normal distribution centered around the provided mean with relation to the standard deviation.

These adjustments maintain the variability of ϵ, allowing it to function as indefinite volatility while still providing some extent of intentional evaluation. Furthermore, ϵ can still theoretically be any value, which is more accurate to genuine stock fluctuations, as any share is capable of drastic variations. However, gradual progression is rendered with more likeliness to ensure that the resulting values are realistic.

4.1.2. Calculation of expected volatility

In order to calculate σ—the representative variable of the expected volatility of a stock—the standard deviation of a share's values is used over a period of time and multiplied by a constant to generate the stock's annualized volatility. This is a commonly used methodology in volatility calculations and is provided by the equation

$$\sigma = s\sqrt{t} \tag{4.3}$$

with s representing the standard deviation of a stock's prices over a time period in the form of a percentage value and t serving as a constant which is provided by the number of trading days present in the given calendar year. For the purposes of this study, the number of trading days present in 2023 is utilized, equating to 250. The generated volatility values are used in the final equations and adjust the model to represent the stochastic nature of the given stock.

4.2. Final Equation

The final equation for this model maintains a similar structure and use of variables to the Reddy-Clinton equation but provides variance in the calculation of those values. Additionally, instead of producing only one iteration of the model, the adjusted equation is performed for 10 separate sequences and the resulting daily outputs are collectively averaged to produce a more accurate figure. This equation is given as

$$S_{t+1} = S_t e^{\frac{(\mu - \frac{\sigma^2}{2})}{N} + \sigma\epsilon\sqrt{\frac{1}{N}}} \tag{4.4}$$

where S_{t+1}, S_t, μ, σ, and ϵ are as defined above. Finally, N represents the number of days that the simulation is presented with, allowing the projections to maintain relevance even for extended durations as the values will not diverge from realistic conditions.

5. Applications

The culmination of these equations provides a representation of a stock's performance over time. This paper records a study in which the performance and accuracy of these equations are evaluated through a comparison of projected stock performance within a simulation and actual values.

5.1. Design of Study

The model was run individually for each stock over the time period from January 1st, 2023, to June 30, 2023, which corresponds with the two most recent fiscal quarters. Additionally, the model functions independently, essentially producing values autonomously for the designated period as opposed to retaining historical data on a frequent basis.

The selected stocks for the deliberation of the model were strategically selected relative to their sector and market cap, constituting a variety of industries and company valuations. The 5 utilized stocks were AAPL, MCD, OXY, PYPL, and SPG, derived from the technology, food, energy, commerce, and real estate sectors, respectively.

After obtaining constant values from the initial date of the simulation, January 1st, and presenting the historical data for these companies in contrast with estimated occurrences, the following results were produced.

5.2. Error Calculations

For the process of acquiring the projection's accuracy, it is helpful to develop a method of quantifying prediction error. For this simulation, the absolute value for the percent error of the projection for each trading day of the study is calculated by the equation

$$\%Error = |\frac{P-R}{R}| \tag{5.1}$$

with P representing the projected value and R representing the real price of the stock. The composite error is then found by averaging the daily percent error values for a stock throughout the duration of the study, formally known as obtaining the Mean Absolute Percentage Error (MAPE). This methodology is similar that which Reddy and Clinton utilized to quantify model success. The equation for this process is given as

$$MAPE = \frac{\sum_{t=1}^{n} |\frac{P-R}{R}|}{n} \tag{5.2}$$

with n defining the number of time units that were executed, t representing time, and the summation rendering a mathematical representation of the combination and averaging of each day's percent error. MAPE serves as the most effective method of error calculation, as instead of rendering the difference in actual and projected values after the duration of the simulation, the holistic accuracy for each output of the projection is accumulated and averaged.

5.3. Utilized constants

In the modified equation, CAPM is calculated using the Russell 1000 Index over the period of 2021-2023 to provide μ, and r_f is provided using the return rate of US Treasury Bonds during the 26-week period prior to the simulation. This is a logically sound method as the Russell 1000 is a common indicator stock that is representative of the general market environment, while the US Treasury Bonds provide a definite return value that generally lacks risk. Effectively, this provides an accurate rendition of these figures relative to the conditions of the present economic climate.

For the utilized model, expected volatility was calculated using the data presented over the 6-month period prior to the execution of the simulation. For the duration of 7/1/22 to 12/31/22 the standard deviation values of the daily closing values of each stock were compiled and evaluated. Afterwards, the resulting deviation figure was multiplied by an annualized constant of $\sqrt{250}$ to achieve the resulting expected volatility, or σ, values.

In the given equation, ϵ was randomly generated from the evaluated normal distribution curve on the MATLAB engine, retrieving a distinct, random value for each date calculated within the difference equation. Although the mean value of the normal distribution set was 0, the standard deviation used to generate ϵ varies for each stock in relation to the provided expected volatilities.

The obtained standard deviation, expected return, and expected volatility values are provided below:

Stock Name	**Deviation** a	**Expected return** μ	**Expected volatility** σ
AAPL	1.11	0.0745	0.356
MCD	0.92	0.0604	0.178
OXY	1.20	0.0852	0.455
PYPL	1.24	0.0749	0.493
SPG	1.06	0.0789	0.313

TABLE 5.1. Calculated constants for simulation

6. Implications

For each stock, the model greatly adhered to the progression of the legitimate values, following similar trajectories for the duration of the period. As anticipated, the model conforms to the theory that semi-random generation is capable of modeling random occurrences.

6.1. Results and Analysis

The modified model can be used to accurately forecast stock prices, regardless of their volatility, market cap, and sector. For AAPL, MCD, OXY, PYPL, and SPG, the acquired MAPE values were 8.26%, 2.26%, 6.30%, 10.93%, and 6.61%, respectively. Previously, Reddy and Clinton delineated that if a stock's MAPE is below 10%, the contributed prediction is characterized as highly accurate (Reddy and Clinton, 2016); fortunately, four of the five stocks examined adhere to this trajectory, with the deviating stock falling a mere 0.93% outside of the given threshold. Additionally, the success of the model was present throughout an extensive period, performing with precision for a duration of 6 months.

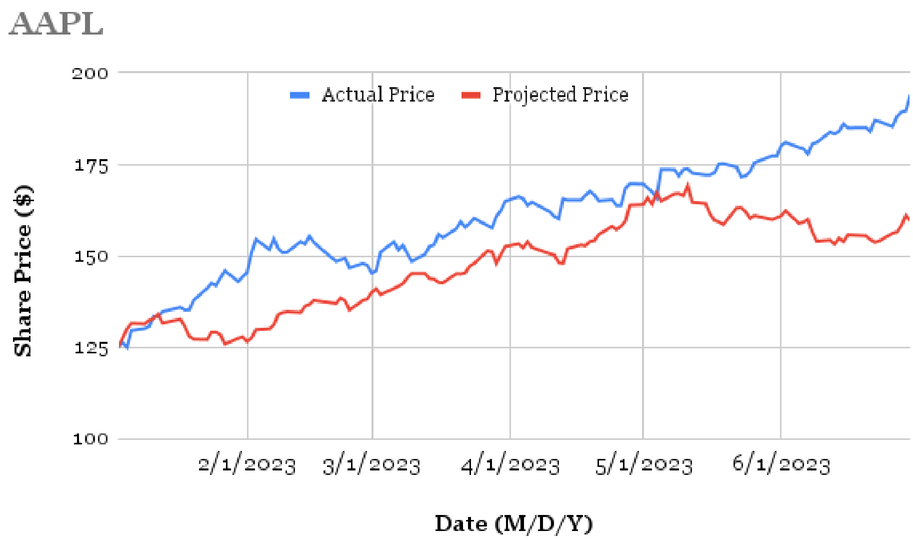

FIGURE 6.1. Actual Price vs. Projected Price of AAPL

The simulated model followed a relatively similar pattern to the stock price's development, further supporting the evaluated success of the modified model as the CAPM efficiently predicted market conditions which were characterized by a specific tier of certainty. However extraneous factors tended to inhibit the success of the model intermittently. Below, graphic representations of the discussed data are attached for each section.

6.1.1. AAPL

AAPL's real stock prices' projection and simulated projection both predicted a steady increase throughout the two fiscal quarters with 93.17% accuracy. However, AAPL experienced extensive positive fluctuation in early May when the company launched its virtual reality headset, Vision Pro. The model projected a steady continuance of the present trajectory, yet this thoroughly publicized occurrence resulted in a heightened positive progression, contributing to the model's difficulty in rendering qualitative and unique factors as opposed to quantitative mathematical trends.

6.1.2. MCD

For MCD, the rendered trajectory was considered most accurate by MAPE evaluations relative to the remaining stocks, providing an incredibly minimal 2.26% error throughout the entire simulated duration. Although Figure 6.3 indicates that the model seemed to somewhat stray from legitimate occurrences in early May due to the development of new marketing initiatives for the company, MCD performed with preeminent accuracy among the presented stocks, largely due to the minimal volatility which encapsulates the share (McDonald's, 2023).

6.1.3. OXY

The initial 35 days of OXY's projection were 87.90% accurate, with the actual share value persisting above the model; however, this variation arose due to supply and demand complications

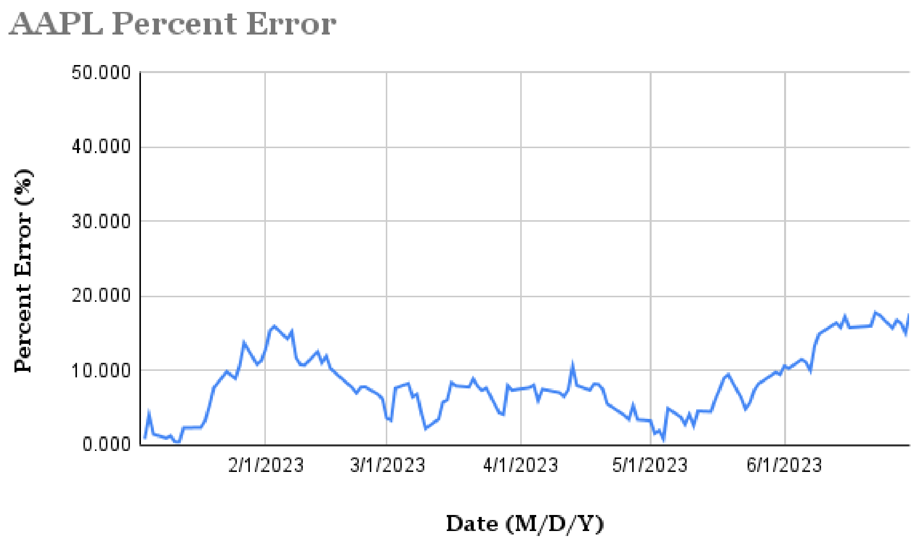

FIGURE 6.2. Percent Error of AAPL

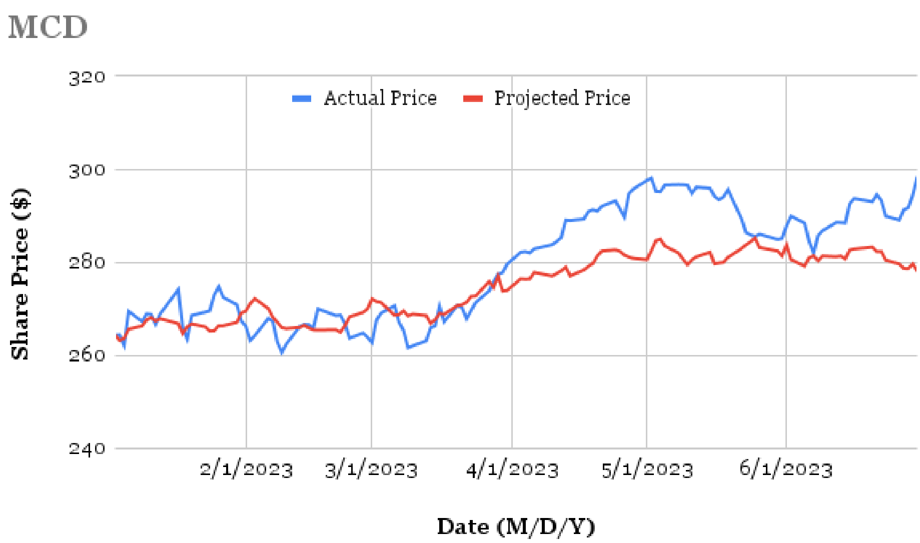

FIGURE 6.3. Actual Price vs. Projected Price of MCD

related to the Russia-Ukraine war (Shilling, 2023). In fact, the modified model correctly predicted that OXY would experience a short-term decline with a 95.43% accuracy for the remaining 4.5 fiscal months of the simulation.

6.1.4. PYPL

PYPL delivered an error of 10.93%, the highest MAPE value of the given stocks. External factors greatly attributed to the negative progression of the share value during the initial months of the simulation. In a world that is gradually emerging from the COVID-19 pandemic, e-commerce

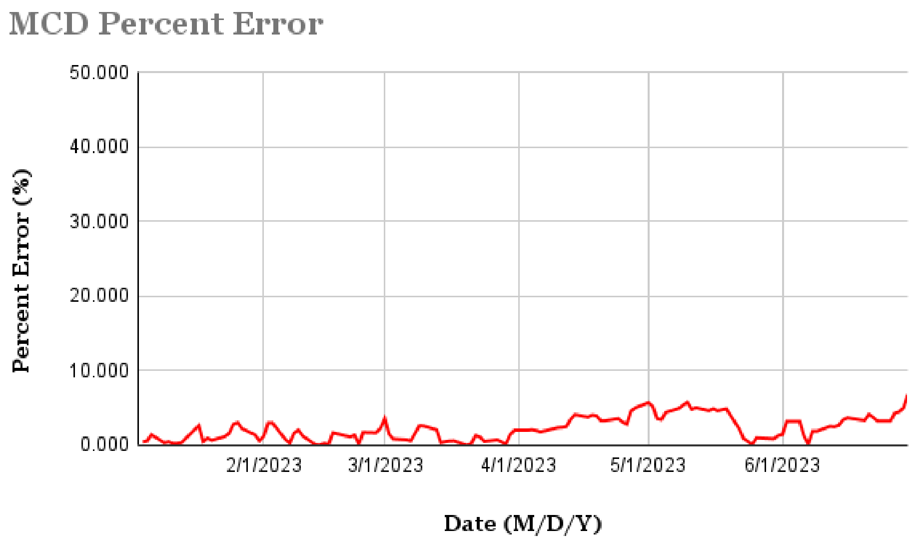

FIGURE 6.4. Percent Error of MCD

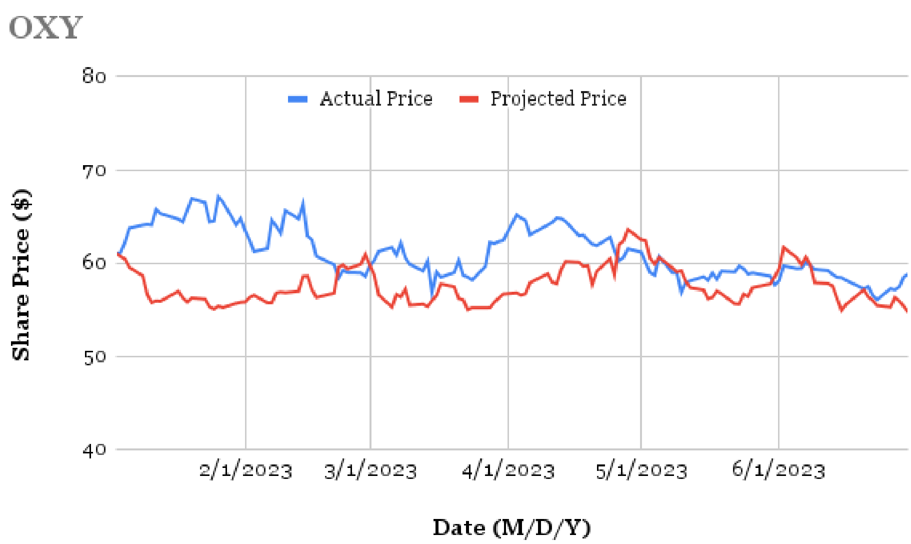

FIGURE 6.5. Actual Price vs. Projected Price of OXY

platforms are facing slight declines in usage (Lee, 2023). Additionally, PYPL faces a multitude of competitors in the online banking industry, contributing to the woes which the stock evidently experienced. However, as witnessed in Figure 6.8, the projection managed to attain near complete accordance with the share value near the latter portion of the simulation, with the final closing values only demonstrating a 4.88% deviation between the actual and predicted results.

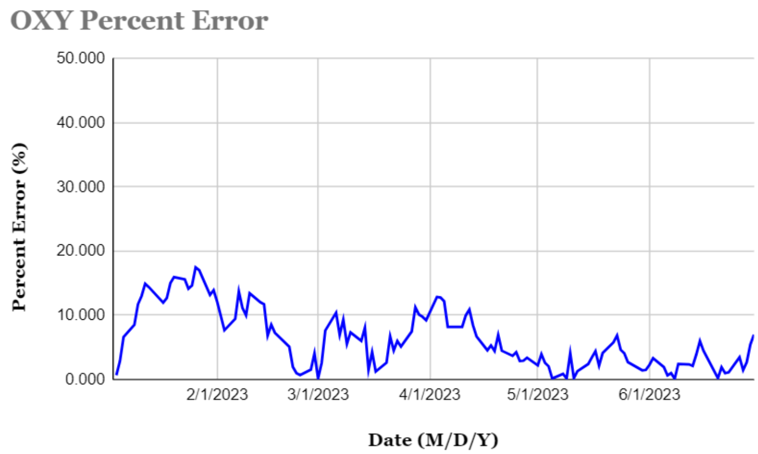

FIGURE 6.6. Percent Error of OXY

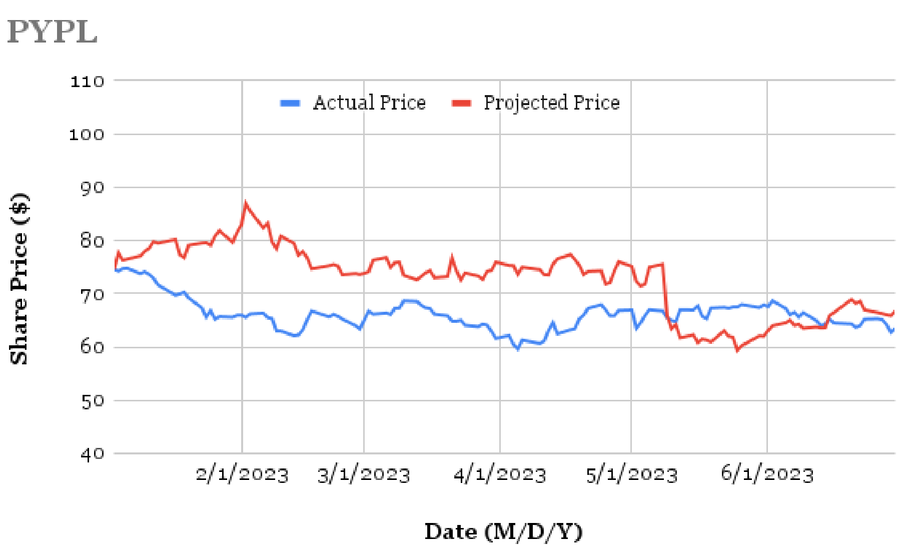

FIGURE 6.7. Actual Price vs. Projected Price of PYPL

6.1.5. SPG

Similarly to OXY, the first 24 fiscal days slightly undervalued SPG's stock price trajectory with a 96.14% accuracy. However, the remaining 98 days of the six-month projection were distinguished by more stable stock prices than the observed fluctuations caused by numerous external factors—the company's risk of bankruptcy, inflation, and declining foothold in a competitive market (Abaterusso, 2022). However, as seen in Figure 6.9, the graph largely remained accurate, following identical increasing and decreasing patterns to the actual share value.

FIGURE 6.8. Percent Error of PYPL

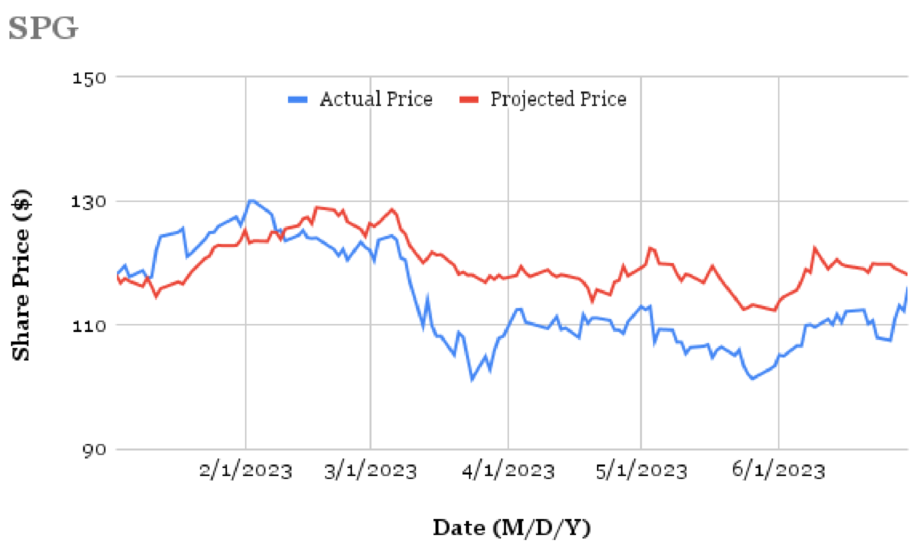

FIGURE 6.9. Actual Price vs. Projected Price of SPG

7. Conclusion

The generated model effectively produces a method of creating a highly accurate rendition of the stock market through the use of existing definitive metrics. The equation proposed in this paper is derived by modifying the Reddy-Clinton equation, primarily by altering the calculation of the ϵ value to more precisely forecast the stock price. Although extraneous real-world occurrences hindered the progression of the model in some scenarios, the issuance of randomized volatility remained impactful in graphically evidencing these occurrences.

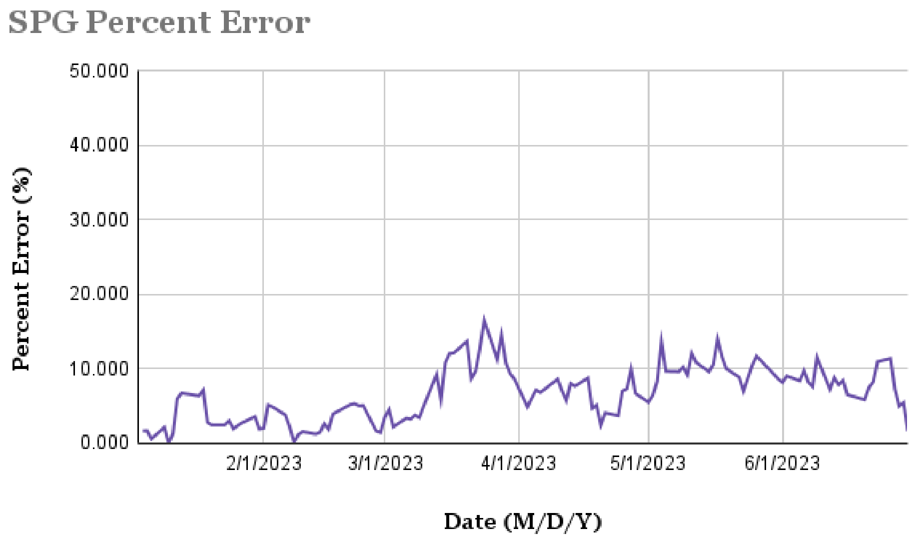

FIGURE 6.10. Percent Error of SPG

The adjustments attributed to the original Reddy-Clinton equation resulted in greater performance of the projected model in relation to actual occurrences. Unexpected volatility was calculated with greater certainty and rational deduction, resulting in a model that functioned with greater authenticity, coherent with the real progression of the share value. Additionally, the utilization of multiple iterations resulted in a similar success, eliminating the possibility for truly random result issuance and allowing for consistent accuracy.

As the time period of the model's execution extends, the model inherently loses accuracy compared to the actual market as it functions independently without retrieving real-world data. To combat this, after a reasonable period of time, such as the utilized half year, the model should be run again with the newly calculated independent variables.

Overall, the model was evidently capable of functioning with exemplary accuracy but faced some difficulty when extraneous real-world occurrences conflicted with the mathematical projection of the given stock. Nevertheless, regardless of these factors, the model continued to provide highly precise extrapolation of the provided factors to effectively define valid predictions with extremely low MAPE values.

Evidently, defining the volatile realm of stock trading is a complex task. However, this model attempts to traverse this convoluted environment to acquire an accurate prediction of a stock's trajectory regardless of sector or market cap. The presence of influencing variables is essentially infinite, yet this model manages to deliver a refined display of primary impacting factors and accurately depict the progression of a given stock as evidenced by the low outputted MAPE values.

For future research, in order to make the randomly generated number proportional to the companies' trends, we plan to transition from a normal distribution curve to a company-specific lognormal distribution curve. This may allow the model to better perceive the trending success of a company and display projections with greater precision.

Acknowledgements

We would like to thank the Summer Ventures Program at the University of North Carolina Wilmington for the opportunity to pursue our interests in economics and the stock market. Their support was essential to the conduction of this research.

References

Abaterusso, A. (2022). Why Simon Property may drop further. https://www.nasdaq.com/articles/why-simon-property-may-drop-further.

Feynman, R. (2013). *The Brownian Movement*, volume 1. California Institute of Technology.

Ibe, O. C. (2013). *Markov processes for stochastic modeling*. Elsevier.

Lee, J. (2023). After hitting new heights during the pandemic, PayPal has struggled in the market. here's what's killing the company's stock growth. http://www.cnbc.com/2023/05/08/heres-whats-killing-paypals-stock-growth.html.

McDonald's (2023). McDonald's strong performance continues. https://corporate.mcdonalds.com/corpmcd/our-stories/article/Q1-2023-earnings.html.

Reddy, K. and Clinton, V. (2016). Simulating stock prices using geometric Brownian motion: Evidence from Australian companies. *Australasian Accounting, Business and Finance Journal*, 10(3).

Shilling, A. (2023). Warren Buffett keeps buying millions of shares of this company despite the stock's decline this year of nearly 10%. Should you join him? https://shorturl.at/cmoDK.

Siegrist, K. (2022). 18.4: Geometric Brownian motion. https://tinyurl.com/yry3xudc.

(A. Kumar) Cary Academy, Cary, NC 27513, USA
Email address: anayk7373@gmail.com

(I. Jamadar) Athens Drive High School, Raleigh, NC 27606, USA
Email address: m.isa4ce@gmail.com

(R. Goel) Cox Mill High School, Concord, NC 28027, USA
Email address: radhikagoel9318@gmail.com

(R. Petluri) Cox Mill High School, Concord, NC 28027, USA
Email address: rcpetluri14@gmail.com

(W. Feng) Department of Mathematics and Statistics, The University of North Carolina at Wilmington, Wilmington, NC 28403, USA
Email address: fengw@uncw.edu