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Note on decomposition of ordered semigroups

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ABSTRACT. Different kinds of decompositions for semigroups and ordered semigroups have been studied by several authors. In this note, we define and study a kind of such a decomposition for ordered semigroups called right o-ideal decomposition. Some properties of it are obtained.

1. Introduction and preliminaries

In general, the problem of decomposing semigroups (resp. po-semigroups) can be approached in different ways depending on the operations (products, unions) on which the decomposition is based (cf. (4; 5; 6; 7; 11; 12)). One of the goals of every decomposition theory for (ordered) semigroups is to describe properties of the whole semigroup by properties of its components and properties of the composition construction. Different kind of decompositions for ordered semigroups has been also studied by several authors (cf. (1; 2; 3; 9; 10)).

By virtue of the importance of decompositions of ordered seigroups, the aim of this note, is to present such a decomposition for ordered semigroups called right (resp. left) o-ideal decomposition.

An ordered semigroup (:po-semigroup) is an ordered set $(S, \leq)$ at the same time a semigroup such that ($\forall a, b, x \in S$), ($a \leq b \Rightarrow xa \leq xb$ and $ax \leq bx$). Let $(S, \cdot, \leq)$ be an ordered semigroup. For a subset H of S, we denote by $(H]$ the subset of S defined by $(H] := \{t \in S | t \leq h$ for some $h \in H\}$. For $A, B \subseteq S$, we denote, $AB := \{ab | a \in A, b \in B\}$. A non-empty subset A of S is called *a left* (resp. *right*) o-ideal of S if (1) $SA \subseteq A$ (resp. $AS \subseteq A$) and (2) If $a \in A$ and $S \ni b \leq a$, then $b \in A$ (or equivalently, $(A] \subseteq A$). A is called an (two-sided) o-ideal of S if it is both a left and a right o-ideal of S. A non-empty subset A of S is called a subsemigroup of S if $A^2 \subseteq A$. It is clear that if A is a subsemigroup of S, then $(A, \cdot, \leq)$ is an ordered semigroup. A mapping f of an ordered semigroup $(S, \cdot, \leq)$ into an ordered semigroup $(H, *, \preceq)$ is called a homomorphism if it satisfies the condition $f(a \cdot b) = f(a) * f(b)$ for each $a, b \in S$ and it is isotone, that is, $a, b \in S$ such that $a \leq b$ implies $f(a) \preceq f(b)$. The mapping f is called an isomorphism, if it is an onto homomorphism and it is a reverse isotone, that is, $a, b \in S$ such that $f(a) \preceq f(b)$ implies $a \leq b$. Recall that any reverse isotone mapping is (1-1).

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2. The main results

In the following definition we define the right o-ideal decomposition of an ordered semigroup S. Some properties of its components are investigated. Analogues results for the left o-ideal decomposition can be obtained.

Definition 2.1. An ordered semigroup S is said to have right o-ideal decomposition if it has two subsemigroups C, T such that $T \cup C = S, T \cap C = \emptyset$, $TC \subseteq T$ and $(T] = T$.

In this section, we study this kind of decomposition of ordered semigroups.

Let C, T be a right o-ideal decomposition of S. We consider the following sets to this decomposition:

$C_C = \{a \in S | a \in C, aT \cap C \neq \emptyset\}$
$C_T = \{a \in S | a \in C, aT \subseteq T\}$
$T_C = \{a \in S | a \in T, Ca \cap C \neq \emptyset\}$
$T_T = \{a \in S | a \in T, Ca \subseteq T\}$.

It is clear that T is a right o-ideal of S. It can be noted that T is an o-ideal if and only if the sets C_C and T_C are empty. In general, the sets C_C, C_T and T_T, if they are nonempty, are subsemigroups of S, and C_T, C_C is a left o-ideal decomposition of C, moreover $T_C C_C \subseteq T_C$. Also, T_C is not necessary a semigroup.

Example 2.2. Let $(S, \cdot, \leq)$ be an ordered semigroup such that the multiplication and partial ordered are defined by:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	c	a
c	a	a	a	a
d	a	d	a	a

$$\leq = \{(a, a), (b, b), (c, c), (d, d), (a, d)\}.$$

Obviously, S is an ordered semigroup with right o-ideal decomposition C, T where $C = \{b\}$ and $T = \{a, c, d\}$.

Example 2.3. (8) Let $(S, \cdot, \leq)$ be an ordered semigroup such that the multiplication and partial ordered are defined by:

$\cdot$	a	b	c	d	e
a	a	b	c	b	b
b	b	b	b	b	b
c	a	b	c	b	b
d	d	b	d	b	b
e	e	e	e	e	e

$$\leq = \{(a, a), (a, c), (b, b), (b, d), (c, c), (d, d), (e, b), (e, d), (e, e)\}.$$

Obviously, S is an ordered semigroup with right o-ideal decomposition C, T where $C = \{a, b, c, d\}$ and $T = \{e\}$.

Example 2.4. Let $S = [0, 1]$ and $*$ the binary operation be defined by $a * b = \min\{a, b\}$ for all $a, b \in S$. Then S is an ordered semigroup with respect to usual ordering. Moreover, if $T = [0, 0.5]$ and $C = (0.5, 1]$, then S is an ordered semigroup with right o-ideal decomposition C, T.

Theorem 2.5. *Let S be an ordered semigroup with right o-ideal decomposition C, T. Then T_T is either empty or is an o-ideal of S.*

Proof. Let assume T_T is non-empty. Let we take $a \in T_T$ and $b \in S = T \cup C$. Firstly, we suppose that $b \in C$. Then $ba \in T$. For all $c \in C$, we have $c(ba) = (cb)a \in T$, thus $ba \in T_T$.

On the other hand, since $TC \subseteq T$, $ab \in T$. Let us suppose that $ab \notin T_T$, that is, $ab \in T_C$. Then $cab \in C$ for some $b \in C$. This implies that $ca \in C$, that is, $a \in T_C$. Since $T \cap C = \emptyset$, this is impossible! Thus $ab \in T_T$ and therefore $CT_T \cup T_TC \subseteq T_T$.

Now let we take $b \in T_C$ and let us suppose that $ba \notin T_T$, that is, $ba \in T_C$. Then we have $cba \in C$ for some $c \in C$ and so $cb \in C$. Therefore, $a \in T_C$ which is impossible since $a \in T_T$. Therefore, $ba \in T_T$.

On the other hand, let us suppose that $ab \notin T_T$, that is, $ab \in T_C$. Then $cab \in C$ for some $c \in C$, and so $ca \in C$, that is, $a \in T_C$ which is impossible since $a \in T_T$. Thus $ab \in T_T$. Therefore, $T_CT_T \cup T_TT_C \subseteq T_T$.

It is obvious that T_T is a subsemigroup, that is, $T_TT_T \subseteq T_T$. From all above, we have $T_TS \cup ST_T \subseteq T_T$ (1).

Now let we prove that $(T_T] \subseteq T_T$. Let $b \in (T_T]$, that is, $b \leq a$ for some $a \in T, Ca \subseteq T$. We have that $Cb \subseteq (Ca] \subseteq (T]$. Hence, since T is a right o-ideal, we have $Cb \subseteq T$. Further, we have $b \in T$ because if $b \in C$, since $Cb \subseteq T$ and C is a subsemigroup of S, then we will have $C \cap T \neq \emptyset$ which is impossible! Therefore, $(T_T] \subseteq T_T$ (2). Finally, from (1) and (2), we have T_T is an o-ideal of S. $\square$

Remark 2.6. We can provide a shorten proof of the above: It is obvious by definition that T_T can be given in a simpler form: $T_T = \{a \in S | S^1a \subseteq T\}$. This immediately implies that T_T, provided non-empty, is an ideal of the semigroup S contained in the right ideal T. For, if $S^1a \subseteq T$ and $s \in S$, then clearly we have $S^1(sa) = (S^1s)a \subseteq S^1a \subseteq T$, and moreover, we have $S^1(as) = (S^1a)s \subseteq Ts \subseteq T$ since T is a right ideal. Hence T_T is both a left and a right ideal. The rest part of the proof is easy, if we replace C with S^1 and considering that $(T_T] \subseteq (T] = T$.

In the following theorem we show under what condition the set T_C is a subsemigroup of S. Let $a \in T$. We consider the set $C(a) = \{b | b \in C, ba \in C\}$.

Theorem 2.7. *Let S be an ordered semigroup with right o-ideal decomposition C, T. Then T_C is a subsemigroup of S if and only if $C(a)a \cap C(b) \neq \emptyset$, for all $a, b \in T_C$.*

Proof. For $a, b \in T_C$, if $ab \in T_C$, then $c(ab) \in C$ for some $c \in C$. Thus $ca \in C$ and therefore $ca \in C(a)a \cap C(b)$. Conversely, if $c \in C(a)a \cap C(b)$, then $c = da$ for some $d \in C$ and $cb \in C$. Hence $dab \in C$, that is, $ab \in T_C$. $\square$

Let we introduce now an equivalence relation ρ on the subset $\{b | b \in T_C, ab \in C_C\}$ of T_C as follows:

$$b \rho b' \Leftrightarrow \forall a \in C_C, ab = ab', \text{ where } b, b' \in T_C.$$

Let we consider the set $U = C_CT_C \cap C_C$ and denote by $[b]_a$ the equivalence class of a. The following theorem shows that the set of equivalence classes $T(a) = \{[b]_a | b \in T_C, ab \in C_C\}$ is isomorphic with the set U which is proved to be a subsemigroup of C_C.

Theorem 2.8. *Let $(S, \cdot, \leq)$ be an ordered semigroup with right o-ideal decomposition C, T and let $a \in C_C$. The following statements hold:*

(1) the set U is a subsemigroup of C_C.

(2) The operation $\circ$ defined by $[b]_a \circ [b']_a = [bab']_a$ is an associative multiplication on the set $T(a)$.

(3) the mapping $f : (T(a), \circ, \preceq) \longrightarrow (U, \cdot, \leq)$, $[b]_a \mapsto ab$ is an isomorphism, where $[b]_a \preceq [b']_a \Leftrightarrow \exists x \in [b]_a, \exists y \in [b']_a$ such that $ax \leq ay$.

Proof. (1) Let we take $a, a' \in C_C$ and $b, b' \in T_C$ such that $ab, a'b' \in U$. Since C_C is a subsemigroup of S, we have $aba'b' \in C_C$. Let we show now that $aba'b' \in C_C T_C$. For this we have to prove that $aba' \in C_C$. If we suppose that $aba' \in T$, since T is a right o-ideal of S, then it follows that $aba'b' \in T$ which is impossible! Thus $aba' \in C$. Also, $aba' \in C_T$ is impossible, since $aba'b' \in C$. Therefore $aba' \in C_C$ and $aba'b' \in C_C T_C$. Hence $aba'b' \in U$ and thus U is a subsemigroup of C_C.

(2) Let $b, b' \in T_C$ such that $ab, ab' \in C_C$ for $a \in C_C$. We claim that $[b]_a \circ [b']_a$ is a subset of $[b'']_a$ for some $b'' \in T_C$ with $ab'' \in C_C$. Let $b_1 \in [b]_a$ and $b'_1 \in [b']_a$. Then $b_1 ab'_1 \in T$. From $ab_1 ab'_1 \in C_C$ it follows that $b_1 ab'_1 \in T_C$. Let $b'' = bab'$ and $b''_1 = b_1 ab'_1$. Then we have $ab''_1 = ab_1 ab'_1 = abab' = ab''$, that is, $ab'', ab''_1 \in C_C$ and $b''_1 \in [b'']_a$.

(3) It is clear that the mapping f is well-defined and bijective. Further, $f([b]_a \circ [b']_a) = f([bab']_a) = abab' = (ab)(ab') = f([b]_a) f([b']_a)$. Also, f is isotone. Indeed: let $[b]_a, [b']_a \in T(a)$, such that $[b]_a \preceq [b']_a$. This implies that there exist $x \in [b]_a, y \in [b']_a, ax \leq ay$. So hence, $ab \leq ab'$. Therefore $f([b]_a) \preceq f([b']_a)$. This shows that the mapping f is a homomorphism. Moreover, it can be easily seen that f is a reverse isotone. Therefore it is an isomorphism. $\square$

In the following theorem we investigate the extreme cases of U being the empty set or the complete set C_C.

Theorem 2.9. *Let S be an ordered semigroup with a right o-ideal decomposition C, T. If $C_C T_C \neq T_T$, that is, $U = C_C$, then T_C and $C_C \cup T_C$ are subsemigroups of S such that $T_C C_C \subseteq T_C$. If $U = \emptyset$, then T is an ideal of S.*

Proof. If $U = C_C T_C \cap C_C = C_C$, then $C_C \subseteq C_C T_C$. Thus, $C_C = C_C T_C$ implies obviously that T_C and $C_C \cup T_C$ are subsemigroups of S such that $C_C \cap T_C = \emptyset$ and $T_C C_C \subseteq T_C$.

Let we consider the case $U = C_C T_C \cap C_C = \emptyset$. From Theorem 2.7(1), since $U = \emptyset$, we have $C_C = \emptyset$ and clearly it follows that $CT \subseteq T$, that is, T is a left o-ideal. Therefore T is an o-ideal of S. $\square$

In the following theorem we investigate the case when C_C contains a subgroup.

Theorem 2.10. *Let S be an ordered semigroup with a right o-ideal decomposition C, T and let us suppose that S contains a subgroup G such that $G \cap C_C \neq \emptyset$. The following statements hold:*

(1) $G \subseteq C_C$;

(2) if $GT \cap G \neq \emptyset$, then $(T(e), \circ, \preceq) \widetilde{-} (G, \cdot, \leq)$ where e is the identity element of G.

Proof. (1) Let $g \in G \cap C_C$. Then there exists $t \in T$ such that $gt \in C$. Let us suppose that $G \nsubseteq C_C$, that is, there exists $g' \in G$ such that $g'T \subseteq T$. Since G is a group and $C_T C_C \subseteq C_C$, then there exists $g'' \in G$ and $g'' \notin C_T$ such that $g' = g''g$. Therefore, $g'' \in T$ and thus $g' \in T$. G is a group implies that there exists $g_0 \in G$ such that $g'g_0 = g$. Since T is a right ideal, we get $g \in T$ which is impossible! Therefore, $G \subseteq C_C$.

(2) Since $GT \cap G \neq \emptyset$, by (1) there exists $t \in T$ such that $et \in G$. Further, for $g \in G$ we have $gt = get \in G$. If $g' \in G$ such that $g' \neq g$, then $g't = g'(et) \neq g(et) = gt$. This shows that for all $g \in G$, there exists $t \in T$ such that $et = g$. Now, if $et = g$ and $et' = g'$, then $ett' = gt' = get' = gg'$. Therefore, $[t]_e \circ [t']_e = [tt']_e$ and arguing in a similar way as in Theorem 2.7(3), we obtain that $(T(e), \circ, \preceq) \widetilde{-} (G, \cdot, \leq)$. $\square$

An immediate corollary of the above theorem is the following.

Corollary 2.11. Let S be an ordered semigroup with a right o-ideal decomposition C, T such that C is a group and $C_C \neq \emptyset$. Then $C = C_C$ and $(T(e), \circ, \preceq)\widetilde{-}(C, \cdot, \leq)$ where e is the identity element of C.

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