North Carolina Journal of
Mathematics and Statistics
Volume 9, Pages 25–50 (Accepted December 15, 2023, published December 20, 2023)
ISSN 2380-7539

Mixed finite element approximations for Darcy flow of isentropic gases

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ABSTRACT. In this paper, the mixed finite element methods are analyzed for the approximation of the solution of the system of equations that describes the single-phase Darcy flow of isentropic gas in a porous medium. Our numerical approach is based on the mixed finite element method (MFEM) in space, and backward-differences in time. The lowest order Raviart-Thomas elements are used. Within this frame work, we derive error estimates in suitable norms and show the convergence of the scheme. The features of the MFEM, especially of the lowest order Raviart- Thomas elements, are now fully exploited in the proof of convergence. Finally, we give the numerical experiments to confirm the theoretical analysis regarding convergence rates.

1. Introduction

We consider a fluid in porous medium occupying a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$ with sufficiently smooth boundary $\partial\Omega$. Let $\mathbf{x} \in \mathbb{R}^d$, $0 < T < \infty$ and $t \in (0, T]$ be the spatial and time variables respectively. The fluid flow has velocity $\mathbf{v}(\mathbf{x}, t) \in \mathbb{R}^d$, pressure $p(\mathbf{x}, t) \in \mathbb{R}$ and density $\rho(\mathbf{x}, t) \in \mathbb{R}_+$. Ward (1964) established from experimental data that

$$\mathbf{v}(\mathbf{x}, t) = -\frac{\kappa(\mathbf{x})}{\mu(\mathbf{x})}\nabla p(\mathbf{x}, t). \tag{1.1}$$

where μ, κ are, respectively (resp.) absolute viscosity and permeability.

Multiplying both sides of the equation (1.1) to ρ, we find that

$$\rho(\mathbf{x}, t)\mathbf{v}(\mathbf{x}, t) = -\frac{\kappa(\mathbf{x})}{\mu(\mathbf{x})}\rho(\mathbf{x}, t)\nabla p(\mathbf{x}, t). \tag{1.2}$$

For isentropic gases, the constitutive law is

$$p(\mathbf{x}, t) = c\rho^\gamma(\mathbf{x}, t) \quad \text{for some } c, \gamma > 1. \tag{1.3}$$

Then, from (1.2) and (1.3), follows

$$\rho(\mathbf{x}, t)\mathbf{v}(\mathbf{x}, t) = -\frac{\kappa(\mathbf{x})}{\mu(\mathbf{x})}\rho(\mathbf{x}, t)\nabla p(\mathbf{x}, t) = -\frac{\kappa(\mathbf{x})}{\mu(\mathbf{x})}\nabla u(\mathbf{x}, t) \quad \text{with } u(\mathbf{x}, t) = \frac{c\gamma\rho(\mathbf{x}, t)^{\gamma+1}}{\gamma + 1}. \tag{1.4}$$

The continuity equation is

$$\phi(\mathbf{x})\partial_t\rho(\mathbf{x}, t) + \operatorname{div}(\rho(\mathbf{x}, t)\mathbf{v}(\mathbf{x}, t)) = f(\mathbf{x}, t), \tag{1.5}$$

where ϕ is the porosity, f is the external mass flow rate.

Received by the editors August 5, 2023.
2010 *Mathematics Subject Classification.* 65M12, 65M15, 65M60, 35Q35, 76S05.
Key words and phrases. Isentropic gas, Darcy's flow, mixed FEM, nonlinear degenerate parabolic equations, Euler implicit scheme, error analysis.

Rewrite

$$\rho(\mathbf{x},t) = \left(\frac{\gamma+1}{c\gamma}\right)^{\lambda} u(\mathbf{x},t)^{\lambda} \quad \text{with } \lambda = \frac{1}{\gamma+1} \in (0,1). \tag{1.6}$$

Combining (1.5) with relation (1.6), we have

$$\phi(\mathbf{x}) \left(\frac{\gamma+1}{c\gamma}\right)^{\lambda} \partial_t u^{\lambda}(\mathbf{x},t) + \operatorname{div}(\rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)) = f(\mathbf{x},t). \tag{1.7}$$

Then from (1.4) and (1.7) we obtain

$$\begin{aligned} \beta(\mathbf{x})\mathbf{m}(\mathbf{x},t) &= -\nabla u(\mathbf{x},t), \\ \phi(\mathbf{x}) \left(\frac{\gamma+1}{c\gamma}\right)^{\lambda} \partial_t u(\mathbf{x},t)^{\lambda} + \operatorname{div}\, \mathbf{m}(\mathbf{x},t) &= f(\mathbf{x},t), \end{aligned} \tag{1.8}$$

where $\mathbf{m}(\mathbf{x},t) = \rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)$, $\beta(\mathbf{x}) = \frac{\mu(\mathbf{x})}{\kappa(\mathbf{x})}$.

By rescaling the variable $\phi(\mathbf{x}) \to (\frac{\gamma+1}{c\gamma})^{\lambda}\phi(\mathbf{x})$, we obtain system of equations

$$\beta(\mathbf{x})\mathbf{m}(\mathbf{x},t) = -\nabla u(\mathbf{x},t), \tag{1.9}$$

$$\phi(\mathbf{x})\partial_t u(\mathbf{x},t)^{\lambda} + \operatorname{div}\, \mathbf{m}(\mathbf{x},t) = f(\mathbf{x},t). \tag{1.10}$$

Substituting the equation (1.9) into the equation (1.10), we obtain a scalar partial differential equation (PDE) for the density:

$$\phi(\mathbf{x})\partial_t u(\mathbf{x},t)^{\lambda} - \operatorname{div}\left(\beta(\mathbf{x})^{-1}\nabla u(\mathbf{x},t)\right) = f(\mathbf{x},t), \quad (\mathbf{x},t) \in \Omega \times (0,T]. \tag{1.11}$$

The numerical analysis of the degenerate parabolic equation arising in flow in porous media using mixed finite element approximations was first studied in Arbogast et al. (1996). Shortly thereafter, Woodward and Dawson (2000) studied the expanded mixed finite element methods for a nonlinear parabolic equation modeling flow into variably saturated porous media. Galerkin finite element method for a coupled nonlinear degenerate system of advection-diffusion equations were studied in Fadimba and Sharpley (2004); Fadimba (2007) and references therein.

The popular numerical method for modeling flow in porous media is the mixed finite element approximations (e.g., Dawson and Wheeler (1994); Kim and Park (1999); Girault and Wheeler (2008); Pan and Rui (2012)). This method is widely used because of its inherent conservation properties and because it produces accurate flux even for highly homogeneous media with large jumps in the conductivity (permeability) tensor Ewing et al. (1996). Since the pioneering work of Raviart and Thomas (1977), the method has become a standard way of deriving high order conservative approximations. We recommend to the reader Brezzi and Fortin (2012) for general accounts of the mixed method.

In this paper, we analyze the order of convergence for a mixed finite element spatial discretization combined with an implicit Euler discretization in time for the Eq. (1.11). We mention Arbogast et al. (1996); Yotov (1997); Radu et al. (2004); Schneid et al. (2004) for a mixed finite element discretization of (1.11). Specifically, the lowest order Raviart–Thomas finite elements are used, whereas the time discretization is achieved by an Euler implicit scheme. For the spatial discretization, optimal error estimates are obtained in Jäger and Kačur (1995); Arbogast et al. (1996); Schneid et al. (2004). For proving the convergence of the fully discrete scheme, the solution is assumed sufficiently regular. Similar results are obtained in Woodward and Dawson (2000) for an expanded MFEM, where three variables are considered explicitly: the pressure, its gradient and the flux.We also mention the combined finite volume - MFEM approach analyzed in Eymard et al. (2006). There are many results on the convergence analysis of the conformal finite element discretization combined with a one step time discretization. Due to the low global regularity of the solutions of degenerate parabolic problems, (see in Alt Hans W. (1983); Woodward and Dawson (2000)), low order discretization methods are well suited for the numerical approximation of the solution. A similar situation appears for the MFEM. The convergence results for the scalar unknowns obtained by both conformal and mixed methods are comparable. While for conformal approaches also estimates that are pointwise in time are available, the mixed approach is providing valuable information on the approximation of the flux. This is due to the specific nature of the method. For the numerical analysis of conformal discretizations of the Richard's equation in the pressure formulation we refer to Nochetto and Verdi (1988) and in the saturation based formulation to Pop (2002), where both type of degeneracy are allowed but the results do not apply to the fully saturated flow regime. We mention Ebmeyer (1998); Pop and Yong (2002) for the porous medium equation.

The outline of this paper is as follows. In section 2, the mixed continuous variational formulation is stated and the regularity of the solution is discussed. The error estimates for the time discrete scheme are obtained in the next section. The fully discrete scheme is considered in section 4 , where error estimates are derived in terms of the discretization parameters. We point out that all these estimates are obtained under minimal regularity of the solution of the problem (2.1). In section 5, the results of a numerical experiment using the Raviart-Thomas elements of order 0 in the two-dimensions are reported and the conclusions.

Notations: Suppose that Ω is an open, bounded subset of $\mathbb{R}^d$, with $d \geq 1$, and has C^1-boundary $\partial\Omega$. Let $L^p(\Omega)$ be a space of functions for which the p-th power of their absolute value is Lebesgue integrable and $(L^p(\Omega))^d$ the space of d-dimensional vectors which have all components in $L^p(\Omega)$. By $\langle\cdot,\cdot\rangle$ we mean the product of duality pairing between $L^p(\Omega)$ and $L^q(\Omega)$ with $1/p + 1/q = 1$ that is

$$\langle \xi, \eta\rangle = \int_\Omega \xi\eta dx \quad \forall \xi \in L^p(\Omega), \eta \in L^q(\Omega) \text{ or } \langle \boldsymbol{\xi}, \boldsymbol{\eta}\rangle = \int_\Omega \boldsymbol{\xi}\cdot\boldsymbol{\eta} dx \quad \forall \boldsymbol{\xi} \in (L^p(\Omega))^d, \boldsymbol{\eta} \in (L^q(\Omega))^d. \tag{1.12}$$

The notation $\|\cdot\|_{0,p}$ is used to denote both norms $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{(L^p(\Omega))^d}$.

We denote $\|u\|_{L^p(0,T;L^q(\Omega))}, 1 \leq p, q < \infty$ means the mixed Lebesgue norm for a function u while $\|u\|_{L^p(0,T;H^q(\Omega))}, 1 \leq p, q < \infty$ stand for the mixed Sobolev-Lebesgue norm of a function u. For $1 \leq q \leq +\infty$ and m any nonnegative integer, let $W^{m,q}(\Omega) = \big\{u \in L^q(\Omega), D^\alpha u \in L^q(\Omega), |\alpha| \leq m\big\}$ denote a Sobolev space endowed with the norm $\|u\|_{m,q} = \Big(\sum_{|\alpha|\leq m} \|D^\alpha u\|^q_{L^q(\Omega)}\Big)^{\frac{1}{q}}$.
Finally we define $H^m(\Omega) = W^{m,2}(\Omega)$.
Throughout this paper, we use short hand notations, $I = (0, T]$, and $u_0(\cdot) = u(\cdot, 0)$.
Our calculations frequently use the following exponents

$$r = 1 + \lambda \in (1, 2), \quad s = \frac{\lambda + 1}{\lambda} = 1 + \frac{1}{\lambda} \in (2, \infty). \tag{1.13}$$

Throughout this paper, we use $C, C_1, C_2, \ldots$ to denote a generic positive constant whose value may change from place to place but are independent of the parameters of the discretization.

We recall some elementary inequalities that will be used in this paper.

For any $p \geq 0$, $x_1, x_2, \ldots, x_k \geq 0$,

$$\frac{x_1^p + x_2^p + \cdots + x_k^p}{k} \leq (x_1 + x_2 + \cdots + x_k)^p \leq k^{(p-1)^+}(x_1^p + x_2^p + \cdots + x_k^p), \tag{1.14}$$

where $z^+ = \max\{z, 0\}$.

2. The mixed finite element method

In this section, we study a mixed finite element approximation to the initial– boundary value problem (IBVP):

$$\begin{aligned} \beta(\mathbf{x})\mathbf{m}(\mathbf{x},t) &= -\nabla u(\mathbf{x},t) \qquad && (\mathbf{x},t) \in \Omega \times I, \\ \phi(\mathbf{x})\partial_t u(\mathbf{x},t)^\lambda + \nabla \cdot \mathbf{m}(\mathbf{x},t) &= f(\mathbf{x},t) \qquad && (\mathbf{x},t) \in \Omega \times I, \\ u(\mathbf{x},t) &= 0 \qquad && (\mathbf{x},t) \in \partial\Omega \times I, \\ u(\mathbf{x},0) &= u_0(\mathbf{x}) \qquad && \mathbf{x} \in \Omega. \end{aligned} \tag{2.1}$$

Noting that we make assumption that $u(\mathbf{x},0) = u_0(\mathbf{x}) = 0$ for all $\mathbf{x} \in \partial\Omega$.

From now on the following assumptions will be needed

(H1) $\phi, \beta \in L^\infty(\Omega)$ and $0 < \phi_* \leq \phi(\mathbf{x}) \leq \phi^* < \infty, 0 < \beta_* \leq \beta(\mathbf{x}) \leq \beta^* < \infty$.
(H2) $u_0 \geq 0$, $u_0 \in H_0^1(\Omega) \cap L^r(\Omega)$.
(H3) $f \in L^\infty(I, H^{-1}(\Omega)) \cap L^\infty(I, L^s(\Omega))$, the function $\|f\|_{0,s}$ to be Lipschitz continuous in time, i.e., there exists a constant $\mathscr{L} > 0$ such that, for every $0 \leq t_1 \leq t_2 \leq T$,

$$\|f(t_1) - f(t_2)\|_{0,s} \leq \mathscr{L}|t_1 - t_2|. \tag{2.2}$$

Remark 2.1. Due to the maximum principle, the solution of the problem (2.1) is great or equal to zero (see in Pop and Yong (2002); Radu et al. (2004); Ivanov and Jäger (2000)).

The existence, uniqueness and essential bounded for weak solution of (2.1) is studied in many papers Alt Hans W. (1983); Kieu (2020a,b); Otto (1996); Raviart (1970); Raviart and Thomas (1983); Schweizer (2007) and the references therein. In particular the following regularity is proven in Alt Hans W. (1983); Raviart (1970); Kieu (2020a,b)

$$\begin{aligned} & u \in L^\infty(I, H_0^1(\Omega)) \cap L^\infty(I, L^r(\Omega)), \\ & u^\lambda \in L^\infty(I, L^{2/\lambda}(\Omega) \cap L^s(\Omega)), \\ & \partial_t u^\lambda \in L^\infty(I, H^{-1}(\Omega)), \\ & \mathbf{m} := -\nabla u \in L^2(I, (L^2(\Omega))^d), \\ & \nabla \cdot \mathbf{m} \in L^2(I, L^s(\Omega)). \end{aligned} \tag{2.3}$$

Since $\partial_t u^\lambda$ is only in $L^\infty(I, H^{-1}(\Omega))$ in the variation formulation of (2.1), this require that the test functions belong to $H_0^1(\Omega)$. However, the mixed formulation requires the test functions are only in $L^2(\Omega)$. To overcome this difficulty we use an idea first proposed by Nochetto (1985)(also see in Arbogast et al. (1996)), we are justified now in integrating the second equation in time from 0 to t and using the last equation in (2.1) to obtain the equivalent distributional equation

$$\phi u^\lambda(\mathbf{x},t) + \nabla \cdot \int_0^t \mathbf{m}\, d\tau = \int_0^t f\, d\tau + \phi u_0^\lambda, \quad (\mathbf{x},t) \in \Omega \times I. \tag{2.4}$$

Note that from (2.3) and (2.4) we conclude that

$$\int_0^t \mathbf{m} d\tau \in H^1(I, (L^2(\Omega))^d) \cap L^2(I, W(\mathrm{div}, \Omega)), \tag{2.5}$$

where $W(\mathrm{div},\Omega)=\left\{\mathbf{v}\in (L^2(\Omega))^d, \nabla\cdot\mathbf{v}\in L^s(\Omega)\right\}\subset H(\mathrm{div},\Omega)$. We denote the norm of $W(\mathrm{div},\Omega)$ by $\|\mathbf{v}\|_{W(\mathrm{div},\Omega)}=\|\mathbf{v}\|_{0,2}+\|\nabla\cdot\mathbf{v}\|_{0,s}$. Since $W(\mathrm{div},\Omega)$ is a closed subspace of $(L^2(\Omega))^{d+1}$, it follows that V is a reflexive Banach space; the boundary $\mathbf{v}\cdot\nu|_{\partial\Omega}$ exist and belong to $W^{1/r,s}(\partial\Omega)$ and we have the Green's formula

$$\int_\Omega \mathbf{v}\nabla\psi dx+\int_\Omega \psi\nabla\cdot\mathbf{v}dx=\int_{\partial\Omega}\psi\mathbf{v}\cdot\nu d\sigma \tag{2.6}$$

hold for every $\mathbf{v}\in W(\mathrm{div},\Omega)$ and $\psi\in (W(\mathrm{div},\Omega))'$ (see Lemma 3 in Brezzi and Fortin (2012)). We have a variational form for almost each time $t\in\bar{I}$ as follows:

Find $(\mathbf{m},u)\in L^2(I,W(\mathrm{div},\Omega))\times L^\infty(I,H_0^1(\Omega))\cap L^\infty(I,L^r(\Omega))$ such that

$$\langle\beta\mathbf{m},\mathbf{v}\rangle-\langle u,\nabla\cdot\mathbf{v}\rangle=0 \quad \text{for all } \mathbf{v}\in W(\mathrm{div},\Omega), \tag{2.7}$$

$$\left\langle\phi u^\lambda(t),q\right\rangle+\left\langle\nabla\cdot\int_0^t\mathbf{m}d\tau,q\right\rangle=\left\langle\int_0^t fd\tau,q\right\rangle+\left\langle\phi u_0^\lambda,q\right\rangle \quad \text{for all } q\in L^r(\Omega). \tag{2.8}$$

In fact (2.8) holds for every $t\in\bar{I}$, since u is defined for each time t. Moreover we can define $\mathbf{m}$ for each time t by (2.7).

3. The time discretization

We now proceed with the time discretization for problem (2.7)–(2.8), which is achieved by using backward Euler for time-difference discretization. Let N be the positive integer, $t_0=0<t_1<\ldots<t_N=T$ be partition interval $[0,T]$ of N sub-intervals, and let $\tau=t_i-t_{i-1}=T/N$ be the i^{th} time step size, $t_i=i\tau$. For any function φ of time let φ_i denote $\varphi(t_i)$. For a given $i=1,\ldots,N$, we define the time discrete mixed variational problem as follow: Let u_{i-1} be given. Find $(\mathbf{m}_i,u_i)\in W(\mathrm{div};\Omega)\times L^r(\Omega)\equiv V\times Q$ such that

$$\langle\beta\mathbf{m}_i,\mathbf{v}\rangle-\langle u_i,\nabla\cdot\mathbf{v}\rangle=0 \quad \text{for all } \mathbf{v}\in V, \tag{3.1}$$

$$\left\langle\phi(u_i^\lambda-u_{i-1}^\lambda),q\right\rangle+\tau\left\langle\nabla\cdot\mathbf{m}_i,q\right\rangle=\tau\left\langle\overline{f}_i,q\right\rangle \quad \text{for all } q\in Q, \tag{3.2}$$

where $\overline{f}_i=\dfrac{1}{\tau}\displaystyle\int_{t_{i-1}}^{t_i}f(t)dt$, whenever $i=1,\ldots N$. For $i=0$, we take $\overline{f}_0=f(0)$.

Initially we take $u_0=u_0(x)$, and $\langle\beta\mathbf{m}_0,\mathbf{v}\rangle=\langle u_0,\nabla\cdot\mathbf{v}\rangle$, for all $\mathbf{v}\in V$.

3.1. Stability for the time-discrete mixed formulation

In this section we investigate the stability of the time discrete approach

Theorem 3.1. *Let* $(\mathbf{m},u)\in V\times Q$ *solve the problem* (3.1)–(3.2). *Suppose that (H1)–(H3). Then, there exists a positive constant* $\mathcal{C}$ *independent of* τ,N *such that for* τ *sufficiently small such that for any* $i=0,\ldots,N$,

$$\|u_i\|_{0,r}+\left\|u_i^\lambda\right\|_{0,s}+\|u_i\|_{1,2}+\|\mathbf{m}_i\|_{0,2}\le\mathcal{C}. \tag{3.3}$$

Proof. Choosing $\mathbf{v}=\mathbf{m}_i$, and $q=u_i$ in (3.1), respectively in (3.2), adding the resultant equations yields

$$\langle\beta\mathbf{m}_i,\mathbf{m}_i\rangle+\left\langle\phi\frac{u_i^\lambda-u_{i-1}^\lambda}{\tau},u_i\right\rangle=\left\langle\overline{f}_i,u_i\right\rangle. \tag{3.4}$$

By virtue of (A.4), we have

$$\left\langle \phi \frac{u_i^\lambda - u_{i-1}^\lambda}{\tau}, u_i \right\rangle \geq \frac{1}{\tau s} \left(\langle \phi, u_i^r \rangle - \langle \phi, u_{i-1}^r \rangle \right). \tag{3.5}$$

Using Young's inequality and the boundedness of the function ϕ gives

$$\langle \overline{f}_i, u_i \rangle \leq \frac{1}{s} \left\| \overline{f}_i \right\|_{0,s}^s + \frac{1}{r} \left\| u_i \right\|_{0,r}^r \leq \frac{1}{s} \left\| \overline{f}_i \right\|_{0,s}^s + \frac{1}{\phi_* r} \langle \phi, u_i^r \rangle. \tag{3.6}$$

Owing to the fact that $\langle \beta \mathbf{m}_i, \mathbf{m}_i \rangle \geq 0$, (3.4), (3.5) and (3.6), it follows that

$$\langle \phi, u_i^r \rangle - \langle \phi, u_{i-1}^r \rangle \leq \tau \left\| \overline{f}_i \right\|_{0,s}^s + \frac{\tau s}{\phi_* r} \langle \phi, u_i^r \rangle. \tag{3.7}$$

If τ sufficient small so that $\ell\tau = \frac{\tau s}{\phi_* r} = \frac{\tau}{\phi_* \lambda} < 1$, which gives $\tau < \phi_* \lambda$, then

$$\langle \phi, u_i^r \rangle \leq \frac{1}{1 - \ell\tau} \left(\langle \phi, u_{i-1}^r \rangle + \tau \left\| \overline{f}_i \right\|_{0,s}^s \right). \tag{3.8}$$

By induction we find that

$$\langle \phi, u_i^r \rangle \leq (1 - \ell\tau)^{-i} \left(\langle \phi, u_0^r \rangle + \sum_{j=1}^{i} (1 - \ell\tau)^{j-1} \tau \left\| \overline{f}_i \right\|_{0,s}^s \right). \tag{3.9}$$

Note that $(1 - \ell\tau)^{-i} \leq e^{\frac{\ell T}{1 - \ell\tau}} < e^{2\ell T}$ for all $\tau < 1/(2\ell) = \frac{1}{2}\phi_* \lambda$, it follows from above inequality that

$$\phi_* \left\| u_i \right\|_{0,r}^r \leq \langle \phi, u_i^r \rangle \leq e^{2\ell T} \left(\phi^* \left\| u_0 \right\|_{0,r}^r + T \left\| f \right\|_{L^\infty(I,L^s)}^s \right) \leq e^{2\ell T} (\phi^* + T) \left(\left\| u_0 \right\|_{0,r}^r + \left\| f \right\|_{L^\infty(I,L^s)}^s \right). \tag{3.10}$$

This leads to

$$\left\| u_i \right\|_{0,r} \leq C_1, \tag{3.11}$$

where $C_1 = \left[\phi_*^{-1} e^{2\ell T} (\phi^* + T) \left(\left\| u_0 \right\|_{0,r}^r + \left\| f \right\|_{L^\infty(I,L^s)}^s \right) \right]^{1/r}$.

It follows easily that

$$\left\| u_i^\lambda \right\|_{0,s} = \left\| u_i \right\|_{0,r}^\lambda \leq C_1^\lambda. \tag{3.12}$$

Using the test function $q = u_i - u_{i-1}$ in (3.2), we obtain

$$\left\langle \phi \frac{u_i^\lambda - u_{i-1}^\lambda}{\tau}, u_i - u_{i-1} \right\rangle + \langle \nabla \cdot \mathbf{m}_i, u_i - u_{i-1} \rangle = \langle \overline{f}_i, u_i - u_{i-1} \rangle. \tag{3.13}$$

Now taking $\mathbf{v} = \mathbf{m}_i$ at time step i and $i - 1$ in (3.1), we have

$$\langle \beta \mathbf{m}_i, \mathbf{m}_i \rangle - \langle \nabla \cdot \mathbf{m}_i, u_i \rangle = 0, \quad \text{and} \quad \langle \beta \mathbf{m}_{i-1}, \mathbf{m}_i \rangle - \langle \nabla \cdot \mathbf{m}_i, u_{i-1} \rangle = 0, \tag{3.14}$$

which implies that

$$\langle \beta(\mathbf{m}_i - \mathbf{m}_{i-1}), \mathbf{m}_i \rangle = \langle \nabla \cdot \mathbf{m}_i, u_i - u_{i-1} \rangle. \tag{3.15}$$

Combining (3.13) and (3.15) shows that

$$\left\langle \phi \frac{u_i^\lambda - u_{i-1}^\lambda}{\tau}, u_i - u_{i-1} \right\rangle + \langle \beta(\mathbf{m}_i - \mathbf{m}_{i-1}), \mathbf{m}_i \rangle = \langle \overline{f}_i, u_i - u_{i-1} \rangle. \tag{3.16}$$

Summing up this equation for $k = 1, 2, \ldots, i$, $i \leq N$ yields

$$\sum_{k=1}^{i} \left\langle \phi \frac{u_k^\lambda - u_{k-1}^\lambda}{\tau}, u_k - u_{k-1} \right\rangle + \langle \beta(\mathbf{m}_k - \mathbf{m}_{k-1}), \mathbf{m}_k \rangle = \sum_{k=1}^{i} \langle \overline{f}_k, u_k - u_{k-1} \rangle. \tag{3.17}$$

We will estimate (3.17) term by term.

The term on the right hand side of (3.17) are bounded by using Hölder's inequality

$$\begin{aligned}\sum_{k=1}^{i}\left\langle \overline{f}_k, u_k-u_{k-1}\right\rangle &= \left\langle \overline{f}_i, u_i\right\rangle - \left\langle \overline{f}_1, u_0\right\rangle + \sum_{k=1}^{i-1}\left\langle \overline{f}_k-\overline{f}_{k+1}, u_k\right\rangle \\ &\le \left\|\overline{f}_i\right\|_{0,s}\|u_i\|_{0,r} + \left\|\overline{f}_1\right\|_{0,s}\|u_0\|_{0,r} + \sum_{k=1}^{i-1}\left\|\overline{f}_k-\overline{f}_{k+1}\right\|_{0,s}\|u_k\|_{0,r} \\ &\le 2C_1\left(\|f\|_{L^\infty(I,L^s)}+\mathscr{L}T\right).\end{aligned} \tag{3.18}$$

For the last term on the left hand side of (3.17), we use Cauchy's inequality and the boundedness of the function β to obtain

$$\begin{aligned}\sum_{k=1}^{i}\langle\beta(\mathbf{m}_k-\mathbf{m}_{k-1}),\mathbf{m}_k\rangle &= \sum_{k=1}^{i}\langle\beta\mathbf{m}_k,\mathbf{m}_k\rangle - \langle\beta\mathbf{m}_{k-1},\mathbf{m}_k\rangle \\ &\ge \frac{\beta_*}{2}\sum_{k=1}^{i}\|\mathbf{m}_k\|_{0,2}^2-\|\mathbf{m}_{k-1}\|_{0,2}^2 = \frac{\beta_*}{2}(\|\mathbf{m}_i\|_{0,2}^2-\|\mathbf{m}_0\|_{0,2}^2).\end{aligned} \tag{3.19}$$

Due to (A.3), the first term on the left hand side of (3.17) is nonnegative that is

$$\sum_{k=1}^{i}\left\langle \phi\frac{u_k^\lambda-u_{k-1}^\lambda}{\tau}, u_k-u_{k-1}\right\rangle \ge 0. \tag{3.20}$$

Substituting (3.18), (3.19) and (3.20) into (3.17) yields

$$\|\mathbf{m}_i\|_{0,2}^2 \le \|\mathbf{m}_0\|_{0,2}^2 + 2\beta_*^{-1}C_1\left(\|f\|_{L^\infty(I,L^s)}+\mathscr{L}T\right). \tag{3.21}$$

Since the choice of initial value of $\mathbf{m}_0$ satisfies

$$\langle\beta\mathbf{m}_0,\mathbf{v}\rangle = \langle u_0,\nabla\cdot\mathbf{v}\rangle = -\langle\nabla u_0,\mathbf{v}\rangle \quad \text{for all } \mathbf{v}\in V. \tag{3.22}$$

Let choose $\mathbf{v}=\mathbf{m}_0\in V$ we have

$$\beta_*\|\mathbf{m}_0\|_{0,2}^2 \le |\langle\nabla u_0,\mathbf{m}_0\rangle| \le \|\nabla u_0\|_{0,2}\|\mathbf{m}_0\|_{0,2}, \tag{3.23}$$

which shows

$$\|\mathbf{m}_0\|_{0,2}^2 \le \beta_*^{-2}\|\nabla u_0\|_{0,2}^2. \tag{3.24}$$

Inserting (3.24) to (3.21) leads to

$$\|\mathbf{m}_i\|_{0,2} \le C_2, \tag{3.25}$$

where $C_2 = \left[\beta_*^{-2}\|\nabla u_0\|_{0,2}^2 + 2\beta_*^{-1}C_1\left(\|f\|_{L^\infty(I,L^s)}+\mathscr{L}T\right)\right]^{1/2}$.

Again from (3.1), we find that

$$\langle\nabla u_i,\mathbf{v}\rangle = -\langle u_i,\nabla\cdot\mathbf{v}\rangle = -\langle\beta\mathbf{m}_i,\mathbf{v}\rangle \quad \text{for all } \mathbf{v}\in V.$$

Consequently,

$$\|\nabla u_i\|_{0,2} \le \beta^*\|\mathbf{m}_i\|_{0,2}.$$

Thanks to Poincaré inequality,

$$\|u_i\|_{1,2} \le C_p\|\nabla u_i\|_{0,2} \le C_p\beta^*\|\mathbf{m}_i\|_{0,2}. \tag{3.26}$$

The assertion (3.3) follows straighforwardly from (3.11), (3.12), (3.25) and (3.26). □

To prove the convergence of the semidiscrete scheme (3.1)-(3.2) we use the following stability estimates:

Lemma 3.2. *Let* $(\mathbf{m}, u) \in V \times Q$ *solve the problem* (2.7)–(2.8). *Suppose that (H1)–(H3). There exists* $C > 0$ *independent of* τ, N *such that for any* $n = 1, \dots, N$,

$$(i) \qquad \sum_{i=1}^{n} \left\| u_i^\lambda - u_{i-1}^\lambda \right\|_{0,s}^s + \tau \sum_{i=1}^{n} \left\| \mathbf{m}_i - \mathbf{m}_{i-1} \right\|_{0,2}^2 \le C\tau. \tag{3.27}$$

$$(ii) \qquad \tau \sum_{i=1}^{n} \left\| \nabla \cdot \mathbf{m}_i \right\|_{0,2}^2 \le C\tau^{-\frac{2(2-r)}{r}}. \tag{3.28}$$

Proof. **Proof of (i).** We rewrite (3.17) as the form

$$\sum_{i=1}^{n} \left\langle \phi \frac{u_i^\lambda - u_{i-1}^\lambda}{\tau}, u_i - u_{i-1} \right\rangle = \sum_{i=1}^{n} \left\langle \beta(\mathbf{m}_{i-1} - \mathbf{m}_i), \mathbf{m}_i \right\rangle + \left\langle \overline{f}_i, u_i - u_{i-1} \right\rangle. \tag{3.29}$$

Using Young's inequality and (3.3), we have

$$\begin{aligned} \sum_{i=1}^{n} \left\langle \beta(\mathbf{m}_{i-1} - \mathbf{m}_i), \mathbf{m}_i \right\rangle &\le \frac{1}{2} \sum_{i=1}^{n} \left(\left\| \sqrt{\beta} \mathbf{m}_{i-1} \right\|^2 - \left\| \sqrt{\beta} \mathbf{m}_i \right\|^2 \right) \\ &\le \frac{\beta^*}{2} \left(\left\| \mathbf{m}_0 \right\|^2 + \left\| \mathbf{m}_n \right\|^2 \right) \le \beta^* \mathcal{C}^2. \end{aligned} \tag{3.30}$$

Combining (3.29), (3.30) and (3.18) we find that

$$\sum_{i=1}^{n} \left\langle \phi \frac{u_i^\lambda - u_{i-1}^\lambda}{\tau}, u_i - u_{i-1} \right\rangle \le C_3, \tag{3.31}$$

where $C_3 \stackrel{\text{def}}{=\!=} 2C_1 \left(\|f\|_{L^\infty(I, L^s)} + \mathscr{L} T \right) + \beta^* \mathcal{C}^2$.

It follows from (A.2) and (3.31) that

$$\begin{aligned} \frac{\phi_*}{\tau} \sum_{i=1}^{n} \left\| u_i^\lambda - u_{i-1}^\lambda \right\|_{0,s}^s &\le \sum_{i=1}^{n} \left\langle \phi \frac{\left| u_i^\lambda - u_{i-1}^\lambda \right|}{\tau}, \left| u_i^\lambda - u_{i-1}^\lambda \right|^{1/\lambda} \right\rangle \\ &\le \sum_{i=1}^{n} \left\langle \phi \frac{u_i^\lambda - u_{i-1}^\lambda}{\tau}, u_i - u_{i-1} \right\rangle \le C_3, \end{aligned} \tag{3.32}$$

We rewrite (3.29) using (A.11) as follows

$$\begin{aligned} -\sum_{i=1}^{n} \left\langle \phi \frac{u_i^\lambda - u_{i-1}^\lambda}{\tau}, u_i - u_{i-1} \right\rangle + \left\langle \overline{f}_i, u_i - u_{i-1} \right\rangle &= \sum_{i=1}^{n} \left\langle \sqrt{\beta}(\mathbf{m}_i - \mathbf{m}_{i-1}), \sqrt{\beta} \mathbf{m}_i \right\rangle \\ &= \frac{1}{2} \left\| \sqrt{\beta} \mathbf{m}_n \right\|_{0,2}^2 - \frac{1}{2} \left\| \sqrt{\beta} \mathbf{m}_0 \right\|_{0,2}^2 + \frac{1}{2} \sum_{i=1}^{n} \left\| \sqrt{\beta}(\mathbf{m}_i - \mathbf{m}_{i-1}) \right\|_{0,2}^2. \end{aligned} \tag{3.33}$$

From (3.33), (3.31), (3.18), (3.3) we find that

$$\begin{aligned}
\beta_* \sum_{i=1}^{n} \|\mathbf{m}_i - \mathbf{m}_{i-1}\|_{0,2}^2 &\le \sum_{i=1}^{n} \left\| \sqrt{\beta}(\mathbf{m}_i - \mathbf{m}_{i-1}) \right\|_{0,2}^2 \\
&\le \left\| \sqrt{\beta}\mathbf{m}_n \right\|_{0,2}^2 + \left\| \sqrt{\beta}\mathbf{m}_0 \right\|_{0,2}^2 + 2 \left| \sum_{i=1}^{n} \left\langle \phi \frac{u_i^\lambda - u_{i-1}^\lambda}{\tau}, u_i - u_{i-1} \right\rangle \right| + 2 \left| \sum_{i=1}^{n} \left\langle \overline{f}_i, u_i - u_{i-1} \right\rangle \right| \\
&\le \beta^* \left(\|\mathbf{m}_n\|_{0,2}^2 + \|\mathbf{m}_0\|_{0,2}^2 \right) + 2 \left| \sum_{i=1}^{n} \left\langle \phi \frac{u_i^\lambda - u_{i-1}^\lambda}{\tau}, u_i - u_{i-1} \right\rangle \right| + 2 \left| \sum_{i=1}^{n} \left\langle \overline{f}_i, u_i - u_{i-1} \right\rangle \right| \le C_4,
\end{aligned} \tag{3.34}$$

where $C_4 \overset{\text{def}}{=\!=} 8 \left[\beta^* \mathcal{C} + C_1 \left(\|f\|_{L^\infty(I,L^s)} + \mathscr{L}T \right) + \beta^* \mathcal{C}^2 \right]$.

The assertion (3.27) follows from (3.32) and (3.34).

Proof of (ii). Since $r < 2 < s$, $L^s(\Omega) \subset L^2(\Omega) \subset L^r(\Omega)$. Using test function $q = \nabla \cdot \mathbf{m}_i \in L^s(\Omega) \subset L^r(\Omega)$ in (3.2), we find that

$$\left\langle \phi(u_i^\lambda - u_{i-1}^\lambda), \nabla \cdot \mathbf{m}_i \right\rangle + \tau \|\nabla \cdot \mathbf{m}_i\|_{0,2}^2 = \tau \left\langle \overline{f}_i, \nabla \cdot \mathbf{m}_i \right\rangle . \tag{3.35}$$

Thanks to the use of Hölder's inequality yields

$$\begin{aligned}
\tau \|\nabla \cdot \mathbf{m}_i\|_{0,2}^2 &\le \left(\tau \left\|\overline{f}_i\right\|_{0,s} + \phi^* \left\|u_i^\lambda - u_{i-1}^\lambda\right\|_{0,s} \right) \|\nabla \cdot \mathbf{m}_i\|_{0,r} \\
&\le C \left(\tau \left\|\overline{f}_i\right\|_{0,s} + \phi^* \left\|u_i^\lambda - u_{i-1}^\lambda\right\|_{0,s} \right) \|\nabla \cdot \mathbf{m}_i\|_{0,2} .
\end{aligned}$$

This gives

$$\|\nabla \cdot \mathbf{m}_i\|_{0,2} \le C\tau^{-1} \left(\tau \left\|\overline{f}_i\right\|_{0,s} + \phi^* \left\|u_i^\lambda - u_{i-1}^\lambda\right\|_{0,s} \right) . \tag{3.36}$$

Summing the above over $i = 1, \dots, n$, $n \le N$ leads to

$$\begin{aligned}
\tau \sum_{i=1}^{n} \|\nabla \cdot \mathbf{m}_i\|_{0,2}^2 &\le \tau^{-1} \sum_{i=1}^{n} \left(\tau \left\|\overline{f}_i\right\|_{0,s} + \phi^* \left\|u_i^\lambda - u_{i-1}^\lambda\right\|_{0,s} \right)^2 \\
&\le 2(1+\phi^*)\tau^{-1} \sum_{i=1}^{n} \left(\tau^2 \left\|\overline{f}_i\right\|_{0,s}^2 + \left\|u_i^\lambda - u_{i-1}^\lambda\right\|_{0,s}^2 \right) .
\end{aligned} \tag{3.37}$$

Furthermore, with $w = 2(s-2)/s = 2(2-r)/r$, we use the Young inequality with $p = s/(s-2)$, $p' = s/2$ to estimate the last term of (3.37)

$$\begin{aligned}
\sum_{i=1}^{n} \left\|u_i^\lambda - u_{i-1}^\lambda\right\|_{0,s}^2 &= \tau^{-w} \sum_{i=1}^{n} \tau^w \left\|u_i^\lambda - u_{i-1}^\lambda\right\|_{0,s}^2 \\
&\le \tau^{-w} \sum_{i=1}^{n} \left(\frac{s-2}{s}\tau^2 + \frac{2}{s} \left\|u_i^\lambda - u_{i-1}^\lambda\right\|_{0,s}^s \right) \\
&\le \frac{\max\{s-2,2\}}{s} \tau^{-w} \left(T\tau + \sum_{i=1}^{n} \left\|u_i^\lambda - u_{i-1}^\lambda\right\|_{0,s}^s \right) .
\end{aligned} \tag{3.38}$$

Due to (3.32),

$$\sum_{i=1}^{n} \left\|u_i^\lambda - u_{i-1}^\lambda\right\|_{0,s}^2 \le \frac{\max\{s-2,2\}}{s} \tau^{-w} \left(T\tau + C_3\phi_*^{-1}\tau \right) \le C\tau^{1-w}. \tag{3.39}$$

Substituting (3.39) to (3.37) gives

$$\begin{aligned}\tau\sum_{i=1}^{n}\|\nabla\cdot\mathbf{m}_i\|_{0,2}^2 &\le 2(1+\phi^*)\tau^{-1}\left(\sum_{i=1}^{n}\tau^2\left\|\overline{f}_i\right\|_{0,s}^s+\tau^{1-w}\right)\\ &\le C\left(T\tau^z\|f\|_{L^\infty(I,L^s)}^s+1\right)\tau^{-w}\\ &\le C\left(T^{z+1}\|f\|_{L^\infty(I,L^s)}^s+1\right)\tau^{-w}\le C\tau^{-w}.\end{aligned} \tag{3.40}$$

This completes the proof. □

3.2. Convergence analysis

In the follows we prove the convergence of the mixed time discrete scheme (3.1)–(3.2). Recalling $u\in L^\infty(I,L^r(\Omega))$ and by (3.3), we have

$$\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\|u(t)-u_i\|_{0,r}^r\,dt\le 2^{r-1}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(\|u(t)\|_{0,r}^r+\|u_i\|_{0,r}^r)dt\le\mathcal{C}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}dt\le\mathcal{C}T. \tag{3.41}$$

Lemma 3.3. *Let* $(\mathbf{m},u)\in V\times Q$ *solve the problem* (2.7)–(2.8)*. Suppose that (H1)–(H3), for any* $n=1,2,\dots,N$ *we have*

$$\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left\|u^\lambda(t)-u_i^\lambda\right\|_{0,s}^s dt+\left\|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(\mathbf{m}(t)-\mathbf{m}_i)dt\right\|_{0,2}^2+\sum_{i=1}^{n}\left\|\int_{t_{i-1}}^{t_i}(\mathbf{m}(t)-\mathbf{m}_i)dt\right\|_{0,2}^2\le C\tau. \tag{3.42}$$

Proof. For any $i=1,2,\dots,N$, (3.1)–(3.2) immediately implies

$$\langle\beta\mathbf{m}_i,\mathbf{v}\rangle-\langle u_i,\nabla\cdot\mathbf{v}\rangle=0\quad\text{for all }\mathbf{v}\in V, \tag{3.43}$$

$$\left\langle\phi(u_i^\lambda-u_0^\lambda),q\right\rangle+\tau\left\langle\nabla\cdot\sum_{k=1}^{i}\mathbf{m}_k,q\right\rangle=\tau\left\langle\sum_{k=1}^{i}\overline{f}_k,q\right\rangle\quad\text{for all }q\in Q. \tag{3.44}$$

Furthermore, (2.7)–(2.8) can be written as

$$\langle\beta\overline{\mathbf{m}}_i,\mathbf{v}\rangle-\langle\overline{u}_i,\nabla\cdot\mathbf{v}\rangle=0\quad\text{for all }\mathbf{v}\in V, \tag{3.45}$$

$$\left\langle\phi(u^\lambda(t_i)-u^\lambda(0)),q\right\rangle+\tau\left\langle\sum_{k=1}^{i}\nabla\cdot\overline{\mathbf{m}}_k,q\right\rangle=\tau\left\langle\sum_{k=1}^{i}\overline{f}_k,q\right\rangle\quad\text{for all }q\in Q. \tag{3.46}$$

Subtracting (3.43) from (3.45) and (3.44) from (3.46) and recall that $u(0)=u_0$ gives

$$\langle\beta(\overline{\mathbf{m}}_i-\mathbf{m}_i),\mathbf{v}\rangle-\langle\overline{u}_i-u_i,\nabla\cdot\mathbf{v}\rangle=0,$$

$$\left\langle\phi(u^\lambda(t_i)-u_i^\lambda),q\right\rangle+\tau\left\langle\nabla\cdot\sum_{k=1}^{i}\overline{\mathbf{m}}_k-\mathbf{m}_k,q\right\rangle=0.$$

Now taking $\mathbf{v} = \tau \sum_{k=1}^{i} \overline{\mathbf{m}}_k - \mathbf{m}_k \in V$ and $q = \overline{u}_i - u_i \in Q$ and adding the resulting yields

$$\left\langle \phi(u^\lambda(t_i) - u_i^\lambda), \overline{u}_i - u_i \right\rangle + \tau \left\langle \beta(\overline{\mathbf{m}}_i - \mathbf{m}_i), \sum_{k=1}^{i} \overline{\mathbf{m}}_k - \mathbf{m}_k \right\rangle = 0. \tag{3.47}$$

Summing the above over $i = 1, \ldots, n$, $n \le N$ leads to

$$\sum_{i=1}^{n} \left\langle \phi(u^\lambda(t_i) - u_i^\lambda), \overline{u}_i - u_i \right\rangle + \tau \sum_{i=1}^{n} \left\langle \beta(\overline{\mathbf{m}}_i - \mathbf{m}_i), \sum_{k=1}^{i} \overline{\mathbf{m}}_k - \mathbf{m}_k \right\rangle = 0. \tag{3.48}$$

We denote by A, B the terms in the above and estimate them separately.

We rewrite A as follows

$$\begin{aligned} A &= \frac{1}{\tau} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left\langle \phi(u^\lambda(t_i) - u^\lambda(t)), u(t) - u_i \right\rangle dt \\ &+ \frac{1}{\tau} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left\langle \phi(u^\lambda(t) - u_i^\lambda), u(t) - u_i \right\rangle dt := A_1 + A_2. \end{aligned} \tag{3.49}$$

For A_2, noting that by (A.2) we have $|u - u_i| \ge |u^\lambda - u_i^\lambda|^{1/\lambda}$, thus

$$\begin{aligned} A_2 &= \frac{1}{\tau} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left\langle \phi|u^\lambda(t) - u_i^\lambda|, |u(t) - u_i| \right\rangle dt \\ &\ge \frac{\phi_*}{\tau} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left\langle |u^\lambda(t) - u_i^\lambda|, |u^\lambda(t) - u_i^\lambda|^{1/\lambda} \right\rangle dt = \frac{\phi_*}{\tau} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left\| u^\lambda(t) - u_i^\lambda \right\|_{0,s}^{s} dt. \end{aligned} \tag{3.50}$$

For A_1, by Hölder's inequality,

$$\begin{aligned} A_1 &= \frac{1}{\tau} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left\langle \int_{t}^{t_i} \phi \partial_t u^\lambda(s) ds, u(t) - u_i \right\rangle dt \\ &\le \frac{1}{\tau} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \int_{t}^{t_i} \left\| \phi \partial_t u^\lambda(s) \right\|_{0,s} \left\| u(t) - u_i \right\|_{0,r} ds dt \\ &\le \frac{1}{\tau} \sum_{i=1}^{n} \left(\int_{t_{i-1}}^{t_i} \left\| \phi \partial_t u^\lambda(s) \right\|_{0,s} ds \right) \left(\int_{t_{i-1}}^{t_i} \left\| u(t) - u_i \right\|_{0,r} dt \right). \end{aligned} \tag{3.51}$$

By Young's inequality, Hölder's inequality, and recalling (3.41) and regularity of $\partial_t u^\lambda$ and u we have

$$\begin{aligned} A_1 &\le \frac{1}{r\tau} \sum_{i=1}^{n} \left(\int_{t_{i-1}}^{t_i} \left\| \phi \partial_t u^\lambda(s) \right\|_{0,s} ds \right)^r + \frac{1}{s\tau} \sum_{i=1}^{n} \left(\int_{t_{i-1}}^{t_i} \left\| u(t) - u_i \right\|_{0,r} dt \right)^s \\ &\le \frac{1}{r} \sum_{i=1}^{n} \left(\int_{t_{i-1}}^{t_i} \left\| \phi \partial_t u^\lambda(s) \right\|_{0,s}^{s} ds \right)^{r/s} + \frac{1}{s} \sum_{i=1}^{n} \left(\int_{t_{i-1}}^{t_i} \left\| u(t) - u_i \right\|_{0,r}^{r} dt \right)^{s/r} \le C. \end{aligned}$$

For B, using the identity (A.10) we rewrite B as

$$B = \tau \sum_{i=1}^{n} \left\langle \beta(\overline{\mathbf{m}}_i - \mathbf{m}_i), \sum_{k=1}^{i} \overline{\mathbf{m}}_k - \mathbf{m}_k \right\rangle$$
$$= \frac{\tau}{2} \left\| \sqrt{\beta} \sum_{i=1}^{n} \overline{\mathbf{m}}_i - \mathbf{m}_i \right\|_{0,2}^2 + \frac{\tau}{2} \sum_{i=1}^{n} \left\| \sqrt{\beta}(\overline{\mathbf{m}}_i - \mathbf{m}_i) \right\|_{0,2}^2 \tag{3.52}$$
$$\geq \frac{\tau\beta_*}{2} \left\| \sum_{i=1}^{n} \overline{\mathbf{m}}_i - \mathbf{m}_i \right\|_{0,2}^2 + \frac{\tau\beta_*}{2} \sum_{i=1}^{n} \|\overline{\mathbf{m}}_i - \mathbf{m}_i\|_{0,2}^2 .$$

Combining (3.48)–(3.52), we find that

$$\frac{\phi_*}{\tau} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|u^\lambda(t) - u_i^\lambda\|_{0,s}^s \, dt + \frac{\tau\beta_*}{2} \left\| \sum_{i=1}^{n} \overline{\mathbf{m}}_i - \mathbf{m}_i \right\|_{0,2}^2 + \frac{\tau\beta_*}{2} \sum_{i=1}^{n} \|\overline{\mathbf{m}}_i - \mathbf{m}_i\|_{0,2}^2 \leq C. \tag{3.53}$$

The result follows straightforwardly by multiplying above by τ and rescaling constants. □

Lemma 3.4. *Assume (H1)–(H3), for any $n = 1, 2, \ldots, N$ we have*

$$\left\| \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} u(t) - u_i \, dt \right\|_{0,r} \leq C \left\| \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \mathbf{m}(t) - \mathbf{m}_i \, dt \right\|_{0,2} . \tag{3.54}$$

Proof. Subtracting (3.43) from (3.45) and adding the resultant equation for $i = 1 \ldots n$ gives

$$\left\langle \sum_{i=1}^{n} \beta(\overline{\mathbf{m}}_i - \mathbf{m}_i), \mathbf{v} \right\rangle - \left\langle \sum_{i=1}^{n} \overline{u}_i - u_i, \nabla \cdot \mathbf{v} \right\rangle = 0 \quad \text{for all } \mathbf{v} \in V. \tag{3.55}$$

Furthermore, by Lemma A.5, there is a $\mathbf{v} \in V$ such that (A.21) holds for

$$w = \left| \sum_{i=1}^{n} \overline{u}_i - u_i \right|^{r-2} \sum_{i=1}^{n} \overline{u}_i - u_i \in L^s(\Omega). \tag{3.56}$$

Taking this $\mathbf{v}$ above yields

$$\left\| \sum_{i=1}^{n} \overline{u}_i - u_i \right\|_{0,r}^r = \left\langle \sum_{i=1}^{n} \beta(\overline{\mathbf{m}}_i - \mathbf{m}_i), \mathbf{v} \right\rangle \leq \beta^* \left\| \sum_{i=1}^{n} \overline{\mathbf{m}}_i - \mathbf{m}_i \right\|_{0,2} \|\mathbf{v}\|_{0,2}$$
$$\leq \beta^* \left\| \sum_{i=1}^{n} \overline{\mathbf{m}}_i - \mathbf{m}_i \right\|_{0,2} \|w\|_{0,s} = \beta^* \left\| \sum_{i=1}^{n} \overline{\mathbf{m}}_i - \mathbf{m}_i \right\|_{0,2} \left\| \sum_{i=1}^{n} \overline{u}_i - u_i \right\|_{0,r}^{r-1} .$$

It follows directly that

$$\left\| \sum_{i=1}^{n} \overline{u}_i - u_i \right\|_{0,r} \leq \beta^* \left\| \sum_{i=1}^{n} \overline{\mathbf{m}}_i - \mathbf{m}_i \right\|_{0,2} . \tag{3.57}$$

The result follows straightforwardly by multiplying the above by τ. □

Summarizing the estimates in Lemma 3.3 and 3.4, we obtain the following

Theorem 3.5. *Assume (H1)–(H3). Let* $(\mathbf{m}, u) \in V \times Q$ *solve problem* (2.7)–(2.8) *and* $(\mathbf{m}_i, u_i) \in V \times Q$ *solve problem* (3.1)–(3.2)*,* $i = 1, \dots, N$*, for any* $n = 1, 2, \dots, N$ *we have*

$$\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left\| u^\lambda(t) - u_i^\lambda \right\|_{0,s}^s dt + \left\| \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} u(t) - u_i \, dt \right\|_{0,r}^2 + \left\| \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (\mathbf{m}(t) - \mathbf{m}_i) dt \right\|_{0,2}^2 + \sum_{i=1}^{n} \left\| \int_{t_{i-1}}^{t_i} (\mathbf{m}(t) - \mathbf{m}_i) dt \right\|_{0,2}^2 \leq C\tau. \tag{3.58}$$

4. The fully discrete mixed finite element discretization

Based on our semidiscrete analysis we derive error estimates for the fully discretization approximation.

Let $\{\mathcal{T}_h\}_h$ be a quasi-regular polygonalization of Ω (by triangles, rectangles, tetrahedron or possibly hexahedron), with $\max_{\tau \in \mathcal{T}_h} \operatorname{diam} \tau \leq h$. The discrete subspace $V_h \times Q_h \subset V \times Q$ is defined as

$$V_h = \{\mathbf{v} \in V; \mathbf{v}|_T \in RT_0(T) \text{ for all } T \in \mathcal{T}_h\}, \tag{4.1}$$

$$Q_h = \{q \in Q; q|_T \in P_0(T) \text{ for all } T \in \mathcal{T}_h\}, \tag{4.2}$$

with $P_0(T)$ denotes the space of constant and $RT_0(T) = (P_0(T))^d + \mathbf{x}P_0(T)$.

The finite element space V_h is the lowest degree element of the Raviart -Thomas space (cf. Ciarlet (1978); Raviart and Thomas (1977)) and Q_h is the space of piecewise constant functions.

For momentum, let $\Pi : V \to V_h$ be the Raviart-Thomas projection Raviart and Thomas (1977), which satisfies

$$\langle \nabla \cdot (\Pi\mathbf{m} - \mathbf{m}), q \rangle = 0 \quad \text{for all } \mathbf{m} \in V, q \in Q_h. \tag{4.3}$$

For pseudo-density, we use the standard L^2-projection operator, see in Ciarlet (1978), $\pi : Q \to Q_h$, satisfying

$$\begin{aligned} (\pi u - u, q) &= 0 \quad \text{for all } u \in Q, q \in Q_h, \\ (\pi u - u, \nabla \cdot \mathbf{m}_h) &= 0 \quad \text{for all } \mathbf{m}_h \in V_h, u \in Q. \end{aligned} \tag{4.4}$$

This projection has well-known approximation properties, e.g. Bramble et al. (2002); Brezzi and Fortin (1991); Johnson and Thomée (1981).

$$\begin{aligned} \|\Pi\mathbf{m} - \mathbf{m}\|_{0,q} &\leq Ch \|\mathbf{m}\|_{1,q} \quad \forall \mathbf{m} \in V \cap (W^{1,q}(\Omega))^d. \\ \|\pi u - u\|_{0,q} &\leq Ch \|u\|_{1,q} \quad q \in [1, \infty], \forall u \in W^{1,q}(\Omega). \end{aligned} \tag{4.5}$$

The two projections π and Π preserve the commuting property $\operatorname{div} \circ \Pi = \pi \circ \operatorname{div} : V \to Q_h$.

With $i = 1, \dots, N$ the fully discrete problems can be defined as follows:

For given $\{\overline{f}_i\}_{i=1}^N$n and $u_{h,i-1}$. Find a pair $(\mathbf{m}_{h,i}, u_{h,i})$ in $V_h \times Q_h$, $i = 1, 2, \dots, N$ such that

$$\langle \beta\mathbf{m}_{h,i}, \mathbf{v} \rangle - \langle u_{h,i}, \nabla \cdot \mathbf{v} \rangle = 0 \quad \text{for all } \mathbf{v} \in V_h, \tag{4.6}$$

$$\left\langle \phi \frac{u_{h,i}^\lambda - u_{h,i-1}^\lambda}{\tau}, q \right\rangle + \langle \nabla \cdot \mathbf{m}_{h,i}, q \rangle = \langle \overline{f}_i, q \rangle \quad \text{for all } q \in Q_h \tag{4.7}$$

with initial choice

$$
\begin{aligned}
\langle u_{h,0}, q\rangle &= \langle \pi u_0, q\rangle \qquad \text{for all } q \in Q_h, \\
\langle \beta \mathbf{m}_{h,0}, \mathbf{v}\rangle &= \langle u_0, \nabla\cdot\mathbf{v}\rangle \qquad \text{for all } \mathbf{v} \in V_h.
\end{aligned}
\tag{4.8}
$$

4.1. Error analysis for fully discrete method

In this section we derive an error estimate for the fully discrete scheme. First, we give some results that are crucial in getting the convergence results.

Lemma 4.1. *Let* $(\mathbf{m}_i, u_i) \in V \times Q$ *solve problem* (3.1)–(3.2) *and* $(\mathbf{m}_{h,i}, u_{h,i}) \in V_h \times Q_h$ *solve the fully discrete finite element approximation* (4.6)–(4.7) *for each time step* i*, with* $i = 1, \ldots, N$*, for any* $n = 1, \ldots, N$ *we have*

$$
\begin{aligned}
&\sum_{i=1}^{n} \left\| u_i^\lambda - u_{h,i}^\lambda \right\|_{0,s}^{s} + \tau \left\| \sum_{i=1}^{n} \Pi\mathbf{m}_i - \mathbf{m}_{h,i} \right\|_{0,2}^{2} + \tau \sum_{i=1}^{n} \left\| \Pi\mathbf{m}_i - \mathbf{m}_{h,i} \right\|_{0,2}^{2} \\
&\qquad\qquad \le C \left(\sum_{i=1}^{n} \left\| \pi u_i - u_i \right\|_{0,r}^{r} + \sum_{i=1}^{n} \left\| \mathbf{m}_i - \Pi\mathbf{m}_i \right\|_{0,2}^{2} \right).
\end{aligned}
\tag{4.9}
$$

Proof. From (4.7), we immediately obtain

$$
\left\langle \phi(u_{h,k}^\lambda - u_{h,0}^\lambda, q\right\rangle + \tau \sum_{i=1}^{k} (\nabla\cdot\mathbf{m}_{h,i}, q) = \tau \sum_{i=1}^{k} \left\langle \overline{f}_i, q\right\rangle \qquad \text{for all } q \in Q_h. \tag{4.10}
$$

Subtracting (4.10) from (3.44), respectively, (4.6) from (3.43) and recalling the definition of projectors (4.3)-(4.4), we find the error equations

$$
\langle \beta(\mathbf{m}_k - \mathbf{m}_{h,k}), \mathbf{v}\rangle - \langle \pi u_k - u_{h,k}, \nabla\cdot\mathbf{v}\rangle = 0 \qquad \text{for all } v \in V_h, \tag{4.11}
$$

$$
\left\langle \phi(u_k^\lambda - u_{h,k}^\lambda), q\right\rangle + \tau \sum_{i=1}^{k} \left\langle \nabla\cdot(\Pi\mathbf{m}_i - \mathbf{m}_{h,i}), q\right\rangle = 0 \qquad \text{for all } q \in Q_h. \tag{4.12}
$$

Now taking $q = \pi u_k - u_{h,k} \in Q_h$ and $\mathbf{v} = \tau \sum_{i=1}^{k} \Pi\mathbf{m}_i - \mathbf{m}_{h,i} \in V_h$ in (4.11), respectively (4.12), adding the resulting and summing up for $k = 1, \ldots, n$ with $n \le N$ gives

$$
\sum_{k=1}^{n} \left\langle \phi(u_k^\lambda - u_{h,k}^\lambda), \pi u_k - u_{h,k}\right\rangle + \tau \sum_{k=1}^{n} \left\langle \beta(\mathbf{m}_k - \mathbf{m}_{h,k}), \sum_{i=1}^{k} \Pi\mathbf{m}_i - \mathbf{m}_{h,i} \right\rangle = 0. \tag{4.13}
$$

Now we proceed by estimate separately the terms in the above, denoted A, B.

We start by writing

$$
\begin{aligned}
A &= \sum_{k=1}^{n} \left\langle \phi(u_k^\lambda - u_{h,k}^\lambda), \pi u_k - u_{h,k}\right\rangle \\
&= \sum_{k=1}^{n} \left\langle \phi(u_k^\lambda - u_{h,k}^\lambda), u_k - u_{h,k}\right\rangle + \sum_{k=1}^{n} \left\langle \phi(u_k^\lambda - u_{h,k}^\lambda), \pi u_k - u_k\right\rangle := A_1 + A_2.
\end{aligned}
\tag{4.14}
$$

The first term above is positive by (A.3). Moreover by (A.2) we have

$$A_1 \geq \phi_* \sum_{k=1}^{n} \left\langle \left|u_k^\lambda - u_{h,k}^\lambda\right|, \left|u_k^\lambda - u_{h,k}^\lambda\right|^{1/\lambda} \right\rangle = \phi_* \sum_{k=1}^{n} \left\|u_k^\lambda - u_{h,k}^\lambda\right\|_{0,s}^s. \tag{4.15}$$

For A_2, using Young's inequality we obtain for any $\varepsilon > 0$,

$$A_2 \leq \varepsilon\phi^* \sum_{k=1}^{n} \left\|u_k^\lambda - u_{h,k}^\lambda\right\|_{0,s}^s + (\varepsilon s)^{-r/s} r^{-1} \phi^* \sum_{k=1}^{n} \|\pi u_k - u_k\|_{0,r}^r. \tag{4.16}$$

We rewrite B as follows

$$\begin{aligned} B = &\sum_{k=1}^{n} \left\langle \beta(\mathbf{m}_k - \Pi\mathbf{m}_k), \tau \sum_{i=1}^{k} \Pi\mathbf{m}_i - \mathbf{m}_{h,i} \right\rangle \\ &+ \sum_{k=1}^{n} \left\langle \beta(\Pi\mathbf{m}_k - \mathbf{m}_{h,k}), \tau \sum_{i=1}^{k} \Pi\mathbf{m}_i - \mathbf{m}_{h,i} \right\rangle := B_1 + B_2. \end{aligned} \tag{4.17}$$

For B_1, we use (A.10) and the boundedness of the function β:

$$B_1 \geq \frac{\tau\beta_*}{2} \left\| \sum_{k=1}^{n} \Pi\mathbf{m}_k - \mathbf{m}_{h,k} \right\|_{0,2}^2 + \frac{\tau\beta_*}{2} \sum_{k=1}^{n} \|\Pi\mathbf{m}_k - \mathbf{m}_{h,k}\|_{0,2}^2. \tag{4.18}$$

For B_2, using Young's inequality and the boundedness of the function β gives

$$B_2 \leq \frac{\beta^*}{2} \sum_{k=1}^{n} \|\mathbf{m}_k - \Pi\mathbf{m}_k\|_{0,2}^2 + \frac{\tau^2\beta^*}{2} \sum_{k=1}^{n} \left\| \sum_{i=1}^{k} \Pi\mathbf{m}_i - \mathbf{m}_{h,i} \right\|_{0,2}^2. \tag{4.19}$$

Combining (4.13), (4.15), (4.16), (4.18) and (4.19), we find that

$$\begin{aligned} &\phi_* \sum_{k=1}^{n} \left\|u_k^\lambda - u_{h,k}^\lambda\right\|_{0,s}^s + \frac{\tau\beta_*}{2} \left\| \sum_{k=1}^{n} \Pi\mathbf{m}_k - \mathbf{m}_{h,k} \right\|_{0,2}^2 + \frac{\tau\beta^*}{2} \sum_{k=1}^{n} \|\Pi\mathbf{m}_k - \mathbf{m}_{h,k}\|_{0,2}^2 \\ &\qquad \leq \varepsilon\phi^* \sum_{k=1}^{n} \left\|u_k^\lambda - u_{h,k}^\lambda\right\|_{0,s}^s + (\varepsilon s)^{-r/s} r^{-1} \phi^* \sum_{k=1}^{n} \|\pi u_k - u_k\|_{0,r}^r \\ &\qquad\qquad + \frac{\beta^*}{2} \sum_{k=1}^{n} \|\mathbf{m}_k - \Pi\mathbf{m}_k\|_{0,2}^2 + \frac{\tau^2\beta^*}{2} \sum_{k=1}^{n} \left\| \sum_{i=1}^{k} \Pi\mathbf{m}_i - \mathbf{m}_{h,i} \right\|_{0,2}^2. \end{aligned} \tag{4.20}$$

Choosing $\varepsilon = \phi_*/(2\phi^*)$ properly leads to

$$\begin{aligned} &\sum_{k=1}^{n} \left\|u_k^\lambda - u_{h,k}^\lambda\right\|_{0,s}^s + \tau \sum_{k=1}^{n} \|\Pi\mathbf{m}_k - \mathbf{m}_{h,k}\|_{0,2}^2 + \tau \left\| \sum_{k=1}^{n} \Pi\mathbf{m}_k - \mathbf{m}_{h,k} \right\|_{0,2}^2 \\ &\qquad \leq C \left(\sum_{k=1}^{n} \|\pi u_k - u_k\|_{0,r}^r + \sum_{k=1}^{n} \|\mathbf{m}_k - \Pi\mathbf{m}_k\|_{0,2}^2 + \tau^2 \sum_{k=1}^{n} \left\| \sum_{i=1}^{k} \Pi\mathbf{m}_i - \mathbf{m}_{h,i} \right\|_{0,2}^2 \right). \end{aligned} \tag{4.21}$$

Dropping the first two terms in previous inequality, applying the discrete Gronwall inequality (A.15) with

$$a_n = \tau \left\| \sum_{k=1}^{n} \Pi \mathbf{m}_k - \mathbf{m}_{h,k} \right\|_{0,2}^{2}, \quad \text{and} \quad b_n = \sum_{k=1}^{n} \|\pi u_k - u_k\|_{0,r}^{r} + \sum_{k=1}^{n} \|\mathbf{m}_k - \Pi \mathbf{m}_k\|_{0,2}^{2} \tag{4.22}$$

we obtain

$$\begin{aligned} \tau \left\| \sum_{k=1}^{n} \Pi \mathbf{m}_k - \mathbf{m}_{h,k} \right\|_{0,2}^{2} &\leq (1 - C\tau)^{-n} \left(\sum_{k=1}^{n} \|\pi u_k - u_k\|_{0,r}^{r} + \sum_{k=1}^{n} \|\mathbf{m}_k - \Pi \mathbf{m}_k\|_{0,2}^{2} \right) \\ &\leq e^{\frac{CT}{1-C\tau}} \left(\sum_{k=1}^{n} \|\pi u_k - u_k\|_{0,r}^{r} + \sum_{k=1}^{n} \|\mathbf{m}_k - \Pi \mathbf{m}_k\|_{0,2}^{2} \right). \end{aligned} \tag{4.23}$$

Substituting back to (4.21), we also find that

$$\sum_{k=1}^{n} \left\| u_k^{\lambda} - u_{h,k}^{\lambda} \right\|_{0,s}^{s} + \tau \sum_{k=1}^{n} \|\Pi \mathbf{m}_k - \mathbf{m}_{h,k}\|_{0,2}^{2} \leq C e^{\frac{CT}{1-C\tau}} \left(\sum_{k=1}^{n} \|\pi u_k - u_k\|_{0,r}^{r} + \sum_{k=1}^{n} \|\mathbf{m}_k - \Pi \mathbf{m}_k\|_{0,2}^{2} \right). \tag{4.24}$$

Finally, (4.9) follows from (4.23)-(4.24). □

The following technical lemma 4.2 is proved in Douglas and Roberts (1985); Thomas (1977).

Lemma 4.2. *There exists a constant $C > 0$ not depending on the mesh size h, such that given an arbitrary $w \in Q_h$. There exists a $\mathbf{v} \in V_h$ such that*

$$\nabla \cdot \mathbf{v} = w \text{ and } \|\mathbf{v}\|_{0,2} \leq C \|\nabla \cdot \mathbf{v}\|_{0,s} \tag{4.25}$$

with $C > 0$ independence of h, w and $\mathbf{v}$.

Lemma 4.3. *Let $(\mathbf{m}_i, u_i) \in V \times Q$ solve problem (3.1)–(3.2) and $(\mathbf{m}_{h,i}, u_{h,i}) \in V_h \times Q_h$ solve the fully discrete finite element approximation (4.6)–(4.7) for each time step i, $i = 1, \dots, N$. For any $n = 1, \dots, N$ we have*

$$\tau \left\| \sum_{i=1}^{n} \pi u_i - u_{h,i} \right\|_{0,r}^{2} \leq C \left(\sum_{i=1}^{n} \|\pi u_i - u_i\|_{0,r}^{r} + \sum_{i=1}^{n} \|\mathbf{m}_i - \Pi \mathbf{m}_i\|_{0,2}^{2} \right). \tag{4.26}$$

Proof. Subtracting (4.6) from (3.1) we obtain

$$\left\langle \sum_{i=1}^{n} \beta(\mathbf{m}_i - \mathbf{m}_{h,i}), \mathbf{v} \right\rangle - \left\langle \sum_{i=1}^{n} \pi u_i - u_{h,i}, \nabla \cdot \mathbf{v} \right\rangle = 0 \quad \text{for all } \mathbf{v} \in V_h. \tag{4.27}$$

Using Lemma 4.2 there exist $\mathbf{v} \in V_h$ such that

$$\nabla \cdot \mathbf{v} = \tau \left| \sum_{i=1}^{n} \pi u_i - u_{h,i} \right|^{r-2} \sum_{i=1}^{n} \pi u_i - u_{h,i} \text{ and } \|\mathbf{v}\| \leq C\tau \left\| \sum_{i=1}^{n} \pi u_i - u_{h,i} \right\|_{0,r}^{r/s}. \tag{4.28}$$

This implies

$$
\begin{aligned}
\tau \left\| \sum_{i=1}^{n} \pi u_i - u_{h,i} \right\|_{0,r}^{r} &= \left\langle \sum_{i=1}^{n} \beta(\mathbf{m}_i - \mathbf{m}_{h,i}), \mathbf{v} \right\rangle \\
&\leq \beta^* \left\| \sum_{i=1}^{n} \mathbf{m}_i - \mathbf{m}_{h,i} \right\|_{0,2} \|\mathbf{v}\|_{0,2} \\
&\leq \beta^* \left\| \sum_{i=1}^{n} \mathbf{m}_i - \mathbf{m}_{h,i} \right\|_{0,2} \tau \left\| \sum_{i=1}^{n} \pi u_i - u_{h,i} \right\|_{0,r}^{r/s} \\
&= \beta^* \left\| \sum_{i=1}^{n} \mathbf{m}_i - \mathbf{m}_{h,i} \right\|_{0,2} \tau \left\| \sum_{i=1}^{n} \pi u_i - u_{h,i} \right\|_{0,r}^{r-1}.
\end{aligned} \tag{4.29}
$$

It follows

$$
\begin{aligned}
\left\| \sum_{i=1}^{n} \pi u_i - u_{h,i} \right\|_{0,r} &\leq \beta^* \left\| \sum_{i=1}^{n} \mathbf{m}_i - \mathbf{m}_{h,i} \right\|_{0,2} \\
&\leq \beta^* \left\| \sum_{i=1}^{n} \mathbf{m}_i - \Pi \mathbf{m}_i \right\|_{0,2} + \beta^* \left\| \sum_{i=1}^{n} \Pi \mathbf{m}_i - \mathbf{m}_{h,i} \right\|_{0,2}.
\end{aligned} \tag{4.30}
$$

Through the use of the inequality (1.14) and (4.9) we get easily,

$$
\begin{aligned}
\tau \left\| \sum_{i=1}^{K} \pi u_i - u_{h,i} \right\|_{0,r}^{2} &\leq 2(\beta^*)^2 \left(\tau \sum_{i=1}^{n} \|\mathbf{m}_i - \Pi \mathbf{m}_i\|_{0,2}^2 + \tau \left\| \sum_{i=1}^{n} \Pi \mathbf{m}_i - \mathbf{m}_{h,i} \right\|_{0,2}^{2} \right) \\
&\leq C \left(\sum_{i=1}^{n} \|\pi u_i - u_i\|_{0,r}^{r} + \sum_{i=1}^{n} \|\mathbf{m}_i - \Pi \mathbf{m}_i\|_{0,2}^2 \right),
\end{aligned} \tag{4.31}
$$

which completes the proof of (4.26). □

The error estimates between the time discrete and the fully discrete solution provided in Lemmas 4.1 and 4.3 can be summarized in the following theorem.

Theorem 4.4. *Let* $(\mathbf{m}_i, u_i) \in V \times Q$ *solve problem* (3.1)–(3.2) *and* $(\mathbf{m}_{h,i}, u_{h,i}) \in V_h \times Q_h$ *solve the fully discrete finite element approximation* (4.6)–(4.7) *for each time step* i, $i = 1, \ldots, N$. *Then, for any* $n = 1, \ldots, N$ *we have*

$$
\begin{aligned}
\sum_{i=1}^{n} \|u_i^\lambda - u_{h,i}^\lambda\|_{0,s}^s + \tau \left\| \sum_{i=1}^{n} \pi u_i - u_{h,i} \right\|_{0,r}^{2} + \tau \left\| \sum_{i=1}^{n} \Pi \mathbf{m}_i - \mathbf{m}_{h,i} \right\|_{0,2}^{2} + \tau \sum_{i=1}^{n} \|\Pi \mathbf{m}_i - \mathbf{m}_{h,i}\|_{0,2}^2 \\
\leq C \left(h^r \sum_{i=1}^{n} \|u_i\|_{1,2}^r + h^2 \sum_{i=1}^{n} \|\mathbf{m}_i\|_{1,2}^2 \right).
\end{aligned}
$$

Proof. By using (4.9), (4.26) and the project estimates (4.5), we have

$$
\begin{aligned}
&\sum_{i=1}^{n} \left\| u_i^\lambda - u_{h,i}^\lambda \right\|_{0,s}^s + \tau \left\| \sum_{i=1}^{n} \pi u_i - u_{h,i} \right\|_{0,r}^2 + \tau \left\| \sum_{i=1}^{n} \Pi \mathbf{m}_i - \mathbf{m}_{h,i} \right\|_{0,2}^2 + \tau \sum_{i=1}^{n} \left\| \Pi \mathbf{m}_i - \mathbf{m}_{h,i} \right\|_{0,2}^2 \\
&\quad \le C \left(\sum_{i=1}^{n} \| \pi u_i - u_i \|_{0,r}^r + \sum_{i=1}^{n} \| \mathbf{m}_i - \Pi \mathbf{m}_i \|_{0,2}^2 \right) \\
&\quad \le C \left(h^r \sum_{i=1}^{n} \| u_i \|_{1,r}^r + h^2 \sum_{i=1}^{n} \| \mathbf{m}_i \|_{1,2}^2 \right) \\
&\quad \le C \left(h^r \sum_{i=1}^{n} \| u_i \|_{1,2}^r + h^2 \sum_{i=1}^{n} \| \mathbf{m}_i \|_{1,2}^2 \right).
\end{aligned}
$$

□

The first term in the above is bounded, as follows from Theorem 3.1. For the last term on the right we make the following additional assumption

(H4) $\mathbf{m}_i \in (H^1(\Omega))^d$ for all $i = 1, \dots N$ and $\tau \sum_{i=1}^{n} \|\mathbf{m}_i\|_{1,2}^2 \le C\tau^{-\frac{2(2-r)}{r}}$.

The assumption above is not too restrictive, since it involves a negative exponent of the time step τ. It is suggested by the estimate (3.28) obtained for $\nabla \cdot \mathbf{m}_i$. Here we assume a similar bound for all partial derivatives of $\mathbf{m}_i$. The H^1-regularity for $\mathbf{m}_i$ in the multi dimensional case is ensured at least for domains with sufficiently smooth boundaries, (see for example Ladyženskaja et al. (1968), Chapter 4).

Using Theorems 4.4 and 3.5, the project estimates (4.5) and stability estimates, we end up with error estimates for the fully discrete mixed finite element scheme.

Theorem 4.5. *Let* $(\mathbf{m}_i, u_i) \in V \times Q$ *solve problem* (3.1)–(3.2) *and* $(\mathbf{m}_{h,i}, u_{h,i}) \in V_h \times Q_h$ *solve the fully discrete finite element approximation* (4.6)–(4.7) *for each time step* i, $i = 1, \dots, N$. *Suppose that (H1)–(H4). Then, there exists a positive constant* C *independent of* h *and* τ *such that for any* $n = 1, \dots, N$,

$$
\begin{aligned}
&\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left\| u^\lambda(t) - u_{h,i}^\lambda \right\|_{0,s}^s dt + \left\| \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (u(t) - u_{h,i}) dt \right\|_{0,r}^2 \\
&\quad + \left\| \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (\mathbf{m}(t) - \mathbf{m}_{h,i}) dt \right\|_{0,2}^2 + \sum_{i=1}^{n} \left\| \int_{t_{i-1}}^{t_i} (\mathbf{m}(t) - \mathbf{m}_{h,i}) dt \right\|_{0,2}^2 \le C \left(\tau + h^2 \tau^{-\frac{2(2-r)}{r}} \right).
\end{aligned}
\tag{4.32}
$$

Proof. Using the triangle inequality, (1.14) and projector (4.4) we find that

$$\begin{aligned}\left\|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(\mathbf{m}(t)-\mathbf{m}_{h,i})dt\right\|_{0,2}^2 &\le 2\left(\left\|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(\mathbf{m}(t)-\mathbf{m}_i)dt\right\|_{0,2}^2+\left\|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(\Pi\mathbf{m}_i-\mathbf{m}_{h,i})dt\right\|_{0,2}^2\right)\\ &= 2\left(\left\|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(\mathbf{m}(t)-\mathbf{m}_i)dt\right\|_{0,2}^2+\tau^2\left\|\sum_{i=1}^{n}\Pi\mathbf{m}_i-\mathbf{m}_{h,i}\right\|_{0,2}^2\right).\end{aligned} \tag{4.33}$$

Similarly, we have

$$\sum_{i=1}^{n}\left\|\int_{t_{i-1}}^{t_i}(\mathbf{m}(t)-\mathbf{m}_{h,i})dt\right\|_{0,2}^2 \le 2\left(\sum_{i=1}^{n}\left\|\int_{t_{i-1}}^{t_i}(\mathbf{m}(t)-\mathbf{m}_i)dt\right\|_{0,2}^2+\tau^2\sum_{i=1}^{n}\|\Pi\mathbf{m}_i-\mathbf{m}_{h,i}\|_{0,2}^2\right). \tag{4.34}$$

$$\left\|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(u(t)-u_{h,i})dt\right\|_{0,r}^2 \le 2^{\frac{2(r-1)}{r}}\left(\left\|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(u(t)-u_i)dt\right\|_{0,r}^2+\tau^2\left\|\sum_{i=1}^{n}\pi u_i-u_{h,i}\right\|_{0,r}^2\right). \tag{4.35}$$

$$\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left\|u^\lambda(t)-u_{h,i}^\lambda\right\|_{0,s}^s dt \le 2^{s-1}\left(\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left\|u^\lambda(t)-u_i^\lambda\right\|_{0,s}^s dt+\tau\sum_{i=1}^{n}\left\|u_i^\lambda-u_{h,i}^\lambda\right\|_{0,s}^s\right). \tag{4.36}$$

It follows from (4.33)–(4.36) that

$$\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left\|u^\lambda(t)-u_{h,i}^\lambda\right\|_{0,s}^s dt+\left\|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(u(t)-u_{h,i})dt\right\|_{0,r}^2 \tag{4.37}$$

$$+\left\|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(\mathbf{m}(t)-\mathbf{m}_{h,i})dt\right\|_{0,2}^2+\sum_{i=1}^{n}\left\|\int_{t_{i-1}}^{t_i}(\mathbf{m}(t)-\mathbf{m}_{h,i})dt\right\|_{0,2}^2 \le C(I+\tau J), \tag{4.38}$$

where

$$\begin{aligned}I=&\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left\|u^\lambda(t)-u_i^\lambda\right\|_{0,s}^s dt+\left\|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}u(t)-u_i\,dt\right\|_{0,r}\\ &+\left\|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(\mathbf{m}(t)-\mathbf{m}_i)dt\right\|_{0,2}^2+\sum_{i=1}^{n}\left\|\int_{t_{i-1}}^{t_i}(\mathbf{m}(t)-\mathbf{m}_i)dt\right\|_{0,2}^2.\end{aligned} \tag{4.39}$$

$$\begin{aligned}J=&\sum_{i=1}^{n}\left\|u_i^\lambda-u_{h,i}^\lambda\right\|_{0,s}^s+\tau\left\|\sum_{i=1}^{n}\pi u_i-u_{h,i}\right\|_{0,r}^2\\ &+\tau\left\|\sum_{i=1}^{n}\Pi\mathbf{m}_i-\mathbf{m}_{h,i}\right\|_{0,2}^2+\tau\sum_{i=1}^{n}\|\Pi\mathbf{m}_i-\mathbf{m}_{h,i}\|_{0,2}^2.\end{aligned} \tag{4.40}$$

According to Theorems 3.5 and 4.4,

$$I \le C\tau, \quad \text{and} \quad J \le C\left(h^r \sum_{i=1}^n \|u_i\|_{1,2}^r + h^2 \sum_{i=1}^n \|\mathbf{m}_i\|_{1,2}^2\right). \tag{4.41}$$

By using stability estimates (3.3), (H4) and Young's inequality with $p = \frac{2}{2-r}$, $q = \frac{2}{r}$, we find that

$$\tau J = \tau h^r \sum_{i=1}^n \|u_i\|_{1,2}^r + h^2\tau \sum_{i=1}^n \|\mathbf{m}_i\|_{1,2}^2 \tag{4.42}$$

$$\le h^r \tau \sum_{i=1}^n \mathcal{C}^r + Ch^2\tau^{-\frac{2(2-r)}{2}} \tag{4.43}$$

$$\le C\left(h^r + h^2\tau^{-\frac{2(2-r)}{2}}\right) \tag{4.44}$$

$$\le C\left(\frac{\tau^{(2-r)p}}{p} + \frac{(h^r\tau^{-(2-r)})^q}{q} + h^2\tau^{-\frac{2(2-r)}{2}}\right) \tag{4.45}$$

$$\le C\left(\tau^2 + h^2\tau^{-\frac{2(2-r)}{2}}\right). \tag{4.46}$$

The result follows straightforward. □

5. Numerical results

In this section we carry out numerical experiments using mixed finite element to solve problem (4.6)–(4.7) in two dimensional region. To test the convergence rates of the proposed algorithm, we choose the true solution of the problem (1.9)–(1.10) by

$$\begin{aligned} u(\mathbf{x}, t) &= e^{-t} \sin x_1 \sin x_2, \\ \mathbf{m}(\mathbf{x}, t) &= \left[-e^{-t} \cos x_1 \sin x_2, -e^{-t} \sin x_1 \cos x_2\right]^T \quad (\mathbf{x}, t) \in \Omega \times (0, T]. \end{aligned} \tag{5.1}$$

where $\Omega = [0, \pi]^2$, and $T > 0$ the final time. For simplicity, we take the functions $\phi(\mathbf{x}) = 1$ and $\beta(\mathbf{x}) = 1$ on Ω. The forcing term f is determined from equation $\partial_t u^\lambda + \nabla \cdot \mathbf{m} = f$. Explicitly, the forcing term f, initial condition and boundary condition accordingly are

$$\begin{aligned} f(\mathbf{x}, t) = -\lambda u^\lambda(\mathbf{x}, t) + 2u(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times [0, T], \\ u_0(\mathbf{x}) = \sin x_1 \sin x_2 \quad \mathbf{x} \in \Omega, \\ u(\mathbf{x}, t) = 0 \quad (\mathbf{x}, t) \in \partial\Omega \times [0, T]. \end{aligned} \tag{5.2}$$

We used FEniCS Logg et al. (2012) to perform our numerical simulations. We divided the square into an $\mathcal{N} \times \mathcal{N}$ mesh of squares, each then subdivided into two right triangles using the RectangleMesh class in FEniCS. The triangularization in region Ω is uniform subdivision in each dimension. The computations are performed for $\lambda = 1/2$, $T = 5$ and $T = 20$. We obtain the convergence rates $r_i = \frac{\ln e_{i-1} - \ln e_i}{\ln h_{i-1} - \ln h_i}$ of finite approximation at seven levels with the discretization parameters $h \in \{\pi/4, \pi/8, \pi/16, \pi/32, \pi/64, \pi/128, \pi/256\}$ (the mesh size is actually $h\sqrt{2}$) respectively.

In the view of Theorem 4.5, the time step is taken $\tau = h^{6/5}$ (by equating exponents in the error bound from (4.32)) to ensure the terms τ and $h^2\tau^{-\frac{2(2-r)}{r}}$ the same order. We compute the errors as given in (4.32). The numerical results are listed in Tables 5.1–5.2 below.

$\mathcal{N}$	h	τ	Error	$\tau + h^{6/5}$	Conv. order
4	1.11072E+00	1.13430E+00	2.20203E-03	2.26859E+00	–
8	5.55360E-01	4.93730E-01	6.44806E-04	9.87460E-01	1.77
16	2.77680E-01	2.14909E-01	3.15522E-04	4.29817E-01	1.03
32	1.38840E-01	9.35444E-02	1.22953E-04	1.870897E-01	1.36
64	6.94201E-02	4.07176E-02	5.32935E-05	8.14352E-02	1.21
128	3.47100E-02	1.77234E-02	2.28386E-05	3.54467E-02	1.22
256	1.73550E-02	7.714537E-03	9.78833E-06	1.54291E-02	1.22

TABLE 5.1. *Numerical results (final time $T = 5$, $\tau = h^{6/5}$).*

$\mathcal{N}$	h	τ	Error	$\tau + h^{6/5}$	Conv. order
4	1.11072E+00	1.134300E+00	3.45737E-04	2.268594E+00	–
8	5.55360E-01	4.93730E-01	1.53201E-04	9.874600E-01	1.18
16	2.77680E-01	2.14909E-01	6.68915E-05	4.298175E-01	1.20
32	1.38840E-01	9.35444E-02	2.58357E-05	1.870887E-01	1.37
64	6.94201E-02	4.07176E-02	8.38373E-06	8.143521E-02	1.62
128	3.47100E-02	1.77234E-02	2.55572E-06	3.544674E-02	1.71
256	1.73550E-02	7.714537E-03	8.37665E-07	1.542907E-02	1.61

TABLE 5.2. *Numerical results (final time $T = 20$, $\tau = h^{6/5}$).*

As shown in the tables 5.1–5.2. the numerical results are confirming the theoretically estimated convergence order of $\tau + h^{6/5}$.

6. Conclusions

In this paper, we have analysed a numerical scheme for Darcy flows of isentropic gas. The spatial discretization is mixed and based on the lowest order Raviart–Thomas finite elements, whereas the time stepping is performed by the Euler implicit method. We have proven the convergence of the scheme by estimating the error in terms of the discretization parameters. The numerical experiments agree with the estimates derived theoretically. Obviously, this method can be expanded to the case of many dimensions easily. There are some open questions including the possible extension of the method to non-Darcy flows.

Acknowledgment

The authors thank the editor and referees for their valuable comments and suggestions which helped us to improve the results of this paper.

Appendix A.

Lemma A.1 (Young's inequality)**.** *Let p, p' any positive real numbers satisfy $1/p + 1/p' = 1$. If a, b are nonnegative real numbers then for any $\varepsilon > 0$,*

$$ab \leq \frac{(\varepsilon a)^p}{p} + \frac{(\varepsilon^{-1} b)^{p'}}{p'}. \tag{A.1}$$

Lemma A.2. *If $\lambda \in (0, 1]$ then for all real numbers $a, b \geq 0$,*

$$(i) \qquad \left|a^\lambda - b^\lambda\right| \leq |a - b|^\lambda. \tag{A.2}$$

$$(ii) \qquad (a^\lambda - b^\lambda)(a - b) \geq \frac{\lambda |a - b|^2}{|a|^{1-\lambda} + |b|^{1-\lambda}}. \tag{A.3}$$

$$(iii) \qquad (a^\lambda - b^\lambda)a \geq \frac{\lambda}{1 + \lambda}\left(a^{1+\lambda} - b^{1+\lambda}\right). \tag{A.4}$$

Proof. i. For $a = b$, $a = 0$ or $b = 0$ the inequality is obvious. W.l.o.g. let $a > b > 0$. Since $0 < 1 - \frac{b}{a}, \frac{b}{a} < 1$ and $0 < \lambda < 1$, we have

$$\left|1 - \left(\frac{b}{a}\right)^\lambda\right| \leq \left|1 - \frac{b}{a}\right| \leq \left|1 - \frac{b}{a}\right|^\lambda. \tag{A.5}$$

The inequality (A.2) follows straightforward from (A.5).

ii. For $a = b$, $a = 0$ or $b = 0$ the inequality is obvious.

For $a, b > 0$, $s \in [0, 1]$ let $\gamma(s) = sa + (1 - s)b$, $k(s) = \gamma(s)^\lambda (a - b)$. Then

$$(a^\lambda - b^\lambda)(a - b) = k(1) - k(0) = \int_0^1 k'(s)ds = \lambda |a - b|^2 \int_0^1 \gamma(s)^{\lambda - 1} ds. \tag{A.6}$$

Note that $(x + y)^p \leq x^p + y^p$ for all $x, y \geq 0$, $0 < p \leq 1$. We have

$$\gamma(s)^{1-\lambda} \leq (sa)^{1-\lambda} + ((1 - s)b)^{1-\lambda} \leq a^{1-\lambda} + b^{1-\lambda}.$$

This implies that

$$(a^\lambda - b^\lambda)(a - b) \geq \lambda |a - b|^2 \int_0^1 \frac{1}{a^{1-\lambda} + b^{1-\lambda}} ds = \frac{\lambda |a - b|^2}{a^{1-\lambda} + b^{1-\lambda}}. \tag{A.7}$$

Therefore, we obtain (A.3).

iii. Using the Young inequality with $p = (\lambda + 1)/\lambda$ and $q = \lambda + 1$, we have

$$b^\lambda a \leq \frac{\lambda}{1 + \lambda} b^{\lambda+1} + \frac{1}{1 + \lambda} a^{\lambda+1}. \tag{A.8}$$

Thus,

$$(a^\lambda - b^\lambda)a \geq a^{\lambda+1} - \frac{\lambda}{1 + \lambda} b^{\lambda+1} - \frac{1}{1 + \lambda} a^{\lambda+1}, \tag{A.9}$$

which proves (A.4) hold true. □

In what follows, we will make use of the elementary results below.

Lemma A.3. *For any vector* $a_k, b_k \in \mathbb{R}^d$, $k = 1, \dots, N$, $d \geq 1$*, we have*

$$(i) \qquad \sum_{n=1}^{N} \left(a_n, \sum_{k=1}^{n} a_k \right) = \frac{1}{2} \left| \sum_{n=1}^{N} a_n \right|^2 + \frac{1}{2} \sum_{n=1}^{N} |a_n|^2. \tag{A.10}$$

$$(ii) \qquad \sum_{n=1}^{N} (a_n - a_{n-1}, a_n) = \frac{1}{2}(|a_N|^2 - |a_0|^2) + \frac{1}{2} \sum_{n=1}^{N} |a_n - a_{n-1}|^2. \tag{A.11}$$

$$(iii) \qquad \sum_{n=1}^{N} (a_n - a_{n-1}, b_n) = a_N b_N - a_0 b_0 - \sum_{n=1}^{N} (b_n - b_{n-1}, a_{n-1}) \,. \tag{A.12}$$

We recall a discrete version of Gronwall Lemma in backward difference form, which is useful later.

Lemma A.4 (Discrete Gronwall inequality)**.** *Assume* $\ell > 0, 1 - \ell\tau > 0$ *and the sequences* $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ *satisfying*

$$a_n \leq b_n + \tau\ell \sum_{i=1}^{n} a_i, \quad n = 1, 2, \dots \tag{A.13}$$

then

$$a_n \leq (1 - \ell\tau)^{-1} b_n + \ell\tau \sum_{i=1}^{n-1} (1 - \ell\tau)^{i-(n+1)} b_i. \tag{A.14}$$

Furthermore, if $\{b_n\}_{n=1}^{\infty}$ *is monotonically increasing then*

$$a_n \leq (1 - \ell\tau)^{-n} b_n. \tag{A.15}$$

Proof. As $n = 1$, we have $a_1 \leq b_1 + \tau\ell a_1$. It follows that

$$a_1 \leq b_1 (1 - \ell\tau)^{-1}, \tag{A.16}$$

which shows (A.14) hold true for $n = 1$.

For $n > 1$, let $S_n = \tau\ell \sum_{i=1}^{n} a_i$, and $\bar{a}_n = (1 - \ell\tau)^n S_n$.
A simple calculation shows that

$$\begin{aligned} \bar{a}_n - \bar{a}_{n-1} &= (1 - \ell\tau)^{n-1} (S_n - S_{n-1} - \ell\tau S_n) \\ &= \ell\tau (1 - \ell\tau)^{n-1} (a_n - S_n) \\ &\leq \ell\tau (1 - \tau)^{n-1} b_n. \end{aligned} \tag{A.17}$$

Summation over n leads to

$$\bar{a}_n \leq \bar{a}_1 + \sum_{i=2}^{n} \ell\tau (1 - \ell\tau)^{i-1} b_i. \tag{A.18}$$

This and (A.16) show that

$$\begin{aligned} S_n &\leq (1 - \ell\tau)^{-n} \left((1 - \ell\tau)\ell\tau a_1 + \sum_{i=2}^{n} \ell\tau (1 - \ell\tau)^{i-1} b_i \right) \\ &\leq (1 - \ell\tau)^{-n} \Big(\ell\tau b_1 + \sum_{i=2}^{n} \ell\tau (1 - \ell\tau)^{i-1} b_i \Big) = (1 - \ell\tau)^{-n} \ell\tau \sum_{i=1}^{n} (1 - \ell\tau)^{i-1} b_i. \end{aligned} \tag{A.19}$$

Noting that

$$a_n \le b_n + S_n \le b_n + (1-\ell\tau)^{-n}\Big((1-\ell\tau)\ell\tau a_1 + \sum_{i=2}^{n} \ell\tau(1-\ell\tau)^{i-1} b_i\Big). \tag{A.20}$$

Hence, (A.14) holds true.

If $\{b_n\}_{n=1}^{\infty}$ is monotonically increasing we use $b_i \le b_n$ in the sum of (A.14) and estimate the remaining sum that is a partial sum of geometric sequence. □

Lemma A.5. *Given $w \in L^s(\Omega)$. There exists a $\mathbf{v} \in V$ such that*

$$\nabla \cdot \mathbf{v} = w \text{ and } \|\mathbf{v}\|_{0,2} \le C \|w\|_{0,s} \tag{A.21}$$

with $C > 0$ independence of w.

Proof. Let $w \in L^s(\Omega)$ and u be the solution of Poison's equation

$$\begin{cases} -\Delta u = w & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{A.22}$$

From Ciarlet (1978) we know that this problem has unique solution $u \in H_0^1(\Omega)$. Testing the above equation by u, integrating over domain Ω and using Green's formula, we have

$$\langle \nabla u, \nabla u \rangle = \langle w, u \rangle. \tag{A.23}$$

Recalling Hölder's inequality, and Poincaré's inequality, we immediately obtain

$$\|\nabla u\|_{0,2}^2 = \langle w, u \rangle \le \|w\|_{0,s} \|u\|_{0,r} \le C \|w\|_{0,s} \|u\|_{0,2} \le C \|w\|_{0,s} \|\nabla u\|_{0,2}, \tag{A.24}$$

which shows that $\|\nabla u\|_{0,2} \le C \|w\|_{0,s}$. The result follows now by taking $\mathbf{v} = -\nabla u \in V$. □

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