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Constructing r-Uniform Hypergraphs with Restricted Clique Numbers

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ABSTRACT. Ramsey theory has posed many interesting questions for graph theorists that have yet to be solved. Many different methods have been used to find Ramsey numbers, though very few are actually known. Because of this, more mathematical tools are needed to prove exact values of Ramsey numbers and their generalizations. Budden, Hiller, Lambert, and Sanford have created a lifting from graphs to 3-uniform hypergraphs that has shown promise in extending known Ramsey results to hypergraphs. This paper will consider another analogous lifting for r-uniform hypergraphs and investigate the lifting of Turán graphs.

1. Introduction

Ramsey theory for graphs began when Ramsey (1930) posed the question "How many people must be gathered to guarantee that there are three mutual acquaintances or three mutual strangers?" This question can be answered by considering an arbitrary red/blue coloring of the edges of a complete graph K_n on n vertices. The vertices represent the people at a gathering with mutual acquaintances connected by red edges and mutual strangers connected by blue edges. One then defines the Ramsey number $R(G_1, G_2)$ to be the least $n \in \mathbb{N}$ such that every red/blue coloring of the edges of K_n contains a red G_1-subgraph (a subgraph isomorphic to G_1) or a blue G_2-subgraph. Ramsey's question is equivalent to finding the value of $R(K_3, K_3)$. The proof that $R(K_3, K_3) = 6$ can be found in many undergraduate textbooks on graph theory and the question even appeared on the Putnam exam in 1953 (Chartrand and Zhang, 2012).

In general, it is very difficult to determine specific Ramsey numbers and few exact values are known. However, many lower and upper bounds are known and the reader is referred to Radziszowski (2014) for an up-to-date listing of known results. Even less understood is the corresponding theory in the setting of r-uniform hypergraphs. An r-uniform hypergraph $H = (V, E)$ consists of a set V of vertices and a set E of hyperedges (r-edges) of unordered r-tuples of vertices. Of course, a 2-uniform hypergraph is just a graph. It is standard to write $V(H)$ in place of V and $E(H)$ in place of E when we wish to emphasize the particular hypergraph being considered.

Recently, Budden et al. described a method of "lifting" a graph to form a 3-uniform hypergraph in such a way as to make the clique number of the resulting hypergraph bounded in terms of the clique number of the underlying graph. The significance of the lifting is that it also preserves complements, allowing one to transfer certain Ramsey results on graphs to corresponding results on 3-uniform hypergraphs.

Our goal in this paper is to consider one possible way in which one can generalize the lifting in Budden et al. to construct r-uniform hypergraphs with bounded clique numbers. Unfortunately,

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we will find that complements are not preserved in general, preventing applications to Ramsey theory. So, we will turn our attention to another application: Turán numbers.

2. Lifting graphs to r-uniform hypergraphs

In order to describe the lifting defined in Budden et al., let $\mathcal{G}_2$ denote the set of all graphs containing at least 3 vertices and $\mathcal{G}_3$ the set of all 3-uniform hypergraphs containing at least 3 vertices. Then the lifting $\varphi : \mathcal{G}_2 \longrightarrow \mathcal{G}_3$ sends a graph $\Gamma \in \mathcal{G}_2$ to the 3-uniform hypergraph $\varphi(\Gamma) \in \mathcal{G}_3$ having vertices $V(\varphi(\Gamma)) = V(\Gamma)$ and hyperedges

$$E(\varphi(\Gamma)) := \{abc \mid \text{exactly one of or all of } ab, bc, ca \in E(\Gamma)\}.$$

It is easily checked that φ preserves complements:

$$\overline{\varphi(\Gamma)} = \varphi(\overline{\Gamma}), \quad \text{for all } \Gamma \in \mathcal{G}_2.$$

Here, the complement $\overline{\Gamma}$ of a graph (or hypergraph) Γ has vertex set $V(\overline{\Gamma}) := V(\Gamma)$ and edge (hyperedge) set given by

$$E(\overline{\Gamma}) := \{e \mid e \notin E(\Gamma)\}.$$

Throughout the rest of this paper, fix $r \geq 3$ and assume that $\mathcal{G}_2$ is the set of all graphs of order at least r. Let $\mathcal{G}_r$ denote the set of all r-uniform hypergraphs of order at least r. For any graph $\Gamma \in \mathcal{G}_2$ and any nonempty subset $S \subseteq V(\Gamma)$, the subgraph induced by S, denoted $\Gamma[S]$, has vertex set S and edge set

$$E(\Gamma[S]) := \{xy \mid x, y \in S \text{ and } xy \in E(\Gamma)\}.$$

Define the lifting

$$\varphi^{(r)} : \mathcal{G}_2 \longrightarrow \mathcal{G}_r$$

to be the map that sends a graph $\Gamma \in \mathcal{G}_2$ to the r-uniform hypergraph having the same vertex set and with $x_1 x_2 \cdots x_r$ forming a hyperedge in $\varphi^{(r)}(\Gamma)$ if and only if $\Gamma[\{x_1, x_2, \ldots, x_r\}]$ is the disjoint union of at most $r-1$ complete subgraphs (including the possibility that it is complete itself). Note that this definition satisfies $\varphi = \varphi^{(3)}$. Following the approach used in Budden et al., we now focus on determining which subgraphs map to complete subhypergraphs.

Theorem 2.1. *Let $\Gamma \in \mathcal{G}_2$, $S \subset V(\Gamma)$ a subset containing at least r elements, and $H := \Gamma[S]$. Then $\varphi^{(r)}(H)$ is complete if and only if H is the disjoint union of at most $r-1$ complete subgraphs.*

Proof. Assume that H is the disjoint union of complete subgraphs $C_1, C_2, \ldots, C_\ell$, where $1 \leq \ell \leq r-1$. Let $x_1, x_2, \ldots, x_r$ be any subset of r distinct vertices in S. If $x_i \in C_i$ and $x_j \in C_j$ with $i \neq j$, then $x_i x_j \notin E(H)$ since C_i and C_j are disconnected. Also, for any two vertices $y, z \in C_i$, $yz \in E(H)$ since C_i is complete. Thus, any collection of r distinct vertices in S is a disjoint union of at most $r-1$ complete subgraphs in H, and $\varphi^{(r)}(H)$ is complete. It remains to be shown that if $\varphi^{(r)}(H)$ is complete, then H is the disjoint union of at most $r-1$ complete subgraphs. Since $\varphi^{(r)}(H)$ is assumed to be complete, it follows that the induced subgraph for every subset of exactly r vertices in H is the disjoint union of at most $r-1$ complete subgraphs. Draw H one vertex at a time, beginning with $k = r$ vertices and including all edges incident with each new vertex and the vertices contained in the previous graph. Let H_k be the graph after k vertices have been drawn. We proceed by induction on k. In the initial case $k = r$, the r vertices lift to a hyperedge, so the preimage is a disjoint union of at most $r-1$ complete subgraphs by definition. Now suppose that for $k \geq r$, H_k is the disjoint union of at most $r-1$ complete subgraphs and consider H_{k+1}, where $V(H_{k+1}) = V(H_k) \cup \{x\}$. Assume that H_k is composed of ℓ disjoint complete subgraphs $C_1, C_2, \ldots, C_\ell$, where $1 \leq \ell \leq r-1$. The first case we consider is when x is is an isolated vertex in H_{k+1}. Then H_{k+1} is the disjoint union of $\ell+1$ complete subgraphs. If $\ell = r-1$, then picking a vertex x_i from each C_i, we find that $xx_1x_2\cdots x_{r-1}$ is not a hyperedge in $\varphi(\Gamma)$, contradicting the assumption that it is complete. Hence, $\ell < r-1$ and we find that H_{k+1} is the disjoint union of at most $r-1$ complete subgraphs. In the remaining cases, x is incident with some vertex in H_k, so H_{k+1} has the same number of components as H_k (or possibly fewer). It remains to be shown that if x is incident with a vertex in C_i, then it must be incident with every vertex in C_i and that x cannot be incident with vertices from more than one copy of component. If x is incident with $x_i \in C_i$ and $x_j \in C_j$ for $i \neq j$, then any subset of r vertices from H_{k+1} that contains x, x_i, and x_j cannot be the disjoint union of complete subgraphs since x_i is not adjacent to x_j. Finally, suppose that the (nonempty) neighborhood of x is entirely contained in C_i for a fixed value of i. If C_i consists of a single vertex, then $\{x\} \cup C_i$ forms a K_2 and H_{k+1} is still the disjoint union of $\ell \leq r-1$ complete subgraphs. Otherwise, C_i must contain at least 2 vertices. Suppose that x is adjacent with $y \in C_i$ and z is some other vertex in C_i. Then any collection of r vertices that includes x, y, and z must be a disjoint union of complete subgraphs, forcing xz to also be an edge. Thus, x must be adjacent to all vertices in C_i. We have shown that H_{k+1} must be the disjoint union of at most $r-1$ complete subgraphs and the same must be true for H. $\square$

Recall that the clique number of a graph Γ, denoted $\omega(\Gamma)$, is the maximum order of a complete subgraph. For an r-uniform hypergraph H, the clique number $\omega(H)$ is analogously defined to be the maximum order of a complete subhypergraph. Using the previous theorem, we are able to construct r-uniform hypergraphs that have bounded clique numbers. We obtain the following corollary relating the clique number for $\Gamma \in \mathcal{G}_2$ to that of its image $\varphi^{(r)}(\Gamma)$.

Corollary 2.2. *For all $\Gamma \in \mathcal{G}_2$ for which the clique number satisfies $\omega(\Gamma) \geq r$, we have that*

$$\omega(\Gamma) \leq \omega(\varphi^{(r)}(\Gamma)) \leq (r-1)\omega(\Gamma).$$

Proof. The first inequality follows from the fact that every clique in Γ of order at least r maps to a clique of the same order in $\varphi^{(r)}(\Gamma)$. The second inequality follows from considering the possibility that Γ could contain $r-1$ disjoint complete subgraphs of order $k \geq r$ (this is a "worst case" scenario). When this occurs and $\omega(\Gamma) = k$, Theorem 2.1 implies that $\omega(\varphi^{(r)}(\Gamma)) = (r-1)k$. $\square$

Unless $r = 3$, the map $\varphi^{(r)}$ does not preserve complements in general:

$$\overline{\varphi^{(r)}(\Gamma)} \neq \varphi^{(r)}(\overline{\Gamma}).$$

For example, if P is a path on four vertices, then $\overline{P}$ is also a path on four vertices and $\varphi^{(4)}$ lifts both P and $\overline{P}$ to empty hypergraphs on four vertices. Thus, our results do not immediately imply lower bounds for two-colored hypergraph Ramsey numbers. Still, being able to construct r-uniform hypergraphs with restricted clique numbers is useful for other aspects of extremal graph theory. In particular, we turn our attention to the determination of Turán numbers.

3. Lifting Turán graphs

Now that we can construct r-uniform hypergraphs with restricted clique numbers, we consider a class of hypergraphs in the image of $\varphi^{(r)}$ in which the clique numbers can be further restricted. For a fixed r-uniform hypergraph H, define the Turán number $\mathrm{ex}(n, H)$ to be the maximum number of hyperedges in an H-free r-uniform hypergraph on n vertices. When $r = 2$, Turán (1941) determined the value of $\mathrm{ex}(n, K_{q+1})$. In this case, the optimal graph $T_q(n)$ is known as a Turán graph. It is a complete q-partite graph that is balanced (the number of vertices in any two partite sets differ by at most 1).

In general, if we write $n = mq + k$, where $0 \leq k < q$, then $T_q(n)$ contains k partite sets of cardinality $\lceil \frac{n}{q} \rceil$ and $q - k$ partite sets of cardinality $\lfloor \frac{n}{q} \rfloor$. Here, $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ are the ceiling and floor functions, respectively. One can show that $T_q(n)$ has size $\lfloor \frac{(q-1)n^2}{2q} \rfloor$ and Turán (1941) proved that $T_q(n)$ is optimal, giving the value of $\mathrm{ex}(n, K_{q+1})$. The determination of $\mathrm{ex}(n, K_{q+1}^{(r)})$ for complete r-uniform hypergraphs has proven to be much more difficult and the reader is referred to Keevash (2011) and Nagle (1999) for an overview of the Turán hypergraph problem.

From Corollary 2.2 we have that

$$q \leq \omega(\varphi^{(r)}(T_q(n))) \leq (r-1)q,$$

but we can actually do much better. Recall that an r-uniform hypergraph is q-partite (with $q \geq r$) if its vertex set can be partitioned into q subsets such that each hyperedge contains at most one vertex from any given partite set.

Theorem 3.1. *The hypergraph $\varphi^{(r)}(T_q(n))$ is the complete r-uniform q-partite hypergraph on n vertices that is balanced. Hence, $\omega(\varphi^{(r)}(T_q(n))) = q$.*

Proof. By construction, for every r-edge $x_1x_2\cdots x_r$ in $\varphi^{(r)}(T_q(n))$, the subgraph $T_q(n)[\{x_1, x_2, \ldots, x_r\}]$ is the disjoint union of at most $r-1$ complete subgraphs. If any distinct $x_i, x_j \in \{x_1, x_2, \ldots, x_r\}$ are in the same partite set, then they must be in different complete subgraphs in $T_q(n)[\{x_1, x_2, \ldots, x_r\}]$. Now suppose that $x_k \in \{x_1, x_2, \ldots, x_r\}$ is in a different partite set than x_i and x_j. Then x_ix_k and x_ix_j are both edges in $T_q(n)$, contradicting the fact that x_i and x_j are in different complete subgraphs of $T_q(n)[\{x_1, x_2, \ldots, x_r\}]$. So, whenever $x_1x_2\cdots x_r$ is a hyperedge in $\varphi^{(r)}(T_q(n))$, all of $x_1, x_2, \ldots, x_r$ are in the same partite set or they are all in different partite sets. The first case cannot occur as the $T_q(n)[\{x_1, x_2, \ldots, x_r\}]$ would be the disjoint union of r copies of K_1. Thus, $x_1, x_2, \ldots, x_r$ are all in different partite sets and $T_q(n)[\{x_1, x_2, \ldots, x_r\}]$ is isomorphic to K_r. On the other hand, $T_q(n)$ is defined to be a complete q-partite graph, so every collection of r vertices from different partite sets forms a K_r. Furthermore, any complete subhypergraph in $\varphi^{(r)}(T_q(n))$ contains at most one vertex from any given partite set and any induced subhypergraph of $\varphi^{(r)}(T_q(n))$ containing vertices from distinct partite sets is necessarily complete. $\square$

Now that we know the clique number of $\varphi^{(r)}(T_q(n))$, we focus on determining its size. Unfortunately, this is very difficult in general. However, in the special case where q divides n, the number of hyperedges in $\varphi^{(r)}(T_q(n))$ is

$$\binom{q}{r}\left(\frac{n}{q}\right)^r = \frac{(q-1)(q-2)\cdots(q-r+1)n^r}{r!q^{r-1}}.$$

Note that when $r = 2$ (viewing $\varphi^{(2)}$ as the identity map), this reduces to $\frac{(q-1)n^2}{2q}$, as expected. Thus, we have shown that when q divides n,

$$\mathrm{ex}(n, K_{q+1}^{(r)}) \geq \frac{(q-1)(q-2)\cdots(q-r+1)n^r}{r!q^{r-1}}.$$

When $r > 2$, this bound is not tight as adding hyperedges to form a $K_{q+1}^{(r)}$ would require that two vertices in the $K_{q+1}^{(r)}$ come from the same partite set and at least $\binom{q-1}{r-2}$ new hyperedges would have to be added. Despite this limitation, concrete bounds for hypergraph Turán numbers are elusive and this application demonstrates the potential of the r-uniform lifting for addressing extremal problems.

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