

Equivalent Formulations of the Bunk Bed Conjecture

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ABSTRACT. A bunk bed graph consists of two isomorphic graphs, called the upper and lower bunks, and some additional edges, called posts; each post connects a vertex in the upper bunk with the corresponding isomorphic vertex in the lower bunk. We assign a probability to each edge with each edge in the upper bunk assigned the same probability as the corresponding isomorphic edge in the lower bunk. The probabilities on the posts are arbitrary. We then form a random spanning subgraph of the bunk bed graph by deleting each edge independently with the prescribed probability. The Bunk Bed Conjecture states that in the random subgraph the probability that a vertex x in the upper bunk is connected to some vertex y in the upper bunk is greater than or equal to the probability that x is connected to z , the isomorphic copy of y in the lower bunk. We show that for each $p \in (0, 1)$ the special case of the Bunk Bed Conjecture in which all edge probabilities are p is equivalent to the full conjecture.

1. Introduction

Let G be a graph and let G_0 and G_1 be two disjoint isomorphic copies of G . Fix graph isomorphisms from G to G_0 and from G to G_1 . If $x \in V(G)$ and $e \in E(G)$, then for $i \in \{0, 1\}$, let x_i and e_i be the isomorphic copies of x and e in G_i with respect to those isomorphisms. Let T be a subset of $V(G)$. The *bunk bed graph* $B = BB(G, T)$ is the graph consisting of G_0 and G_1 together with an edge $\{x_0, x_1\}$ for each $x \in T$. G_0 and G_1 are called the *upper* and *lower bunks* of B . We also define $E_T(B) = \{\{x_0, x_1\} : x \in T\}$ to be the set of *posts* of B .

The terms bunk and post are used because the following "3-dimensional drawing" of B is meant to evoke a bunk bed. Identical drawings of G_0 and G_1 are placed in two horizontal planes with each vertex x_0 of G_0 positioned directly above the corresponding vertex x_1 of G_1 . This drawing of B looks a little like the drawing of a bunk bed with the upper bunk G_0 held above the lower bunk G_1 by the posts in $E_T(B)$. See Figures 1.1 and 1.2 below.

Note that the vertices of the upper bunk of B in Figure 1.2 are v_0, w_0, x_0, y_0, z_0 and those of the lower bunk are v_1, w_1, x_1, y_1, z_1 .

An *edge-probability function* for a graph G is a function $\mathbf{p} : E(G) \rightarrow [0, 1]$. If $B = BB(G, T)$ is a bunk bed graph, we say that $\mathbf{p} : E(B) \rightarrow [0, 1]$ is *symmetric* if for all edges $e \in E(G)$, $\mathbf{p}(e_0) = \mathbf{p}(e_1)$. Note that even if \mathbf{p} is symmetric, this implies no restriction on $\mathbf{p}(e)$ if $e \in E_T(B)$.

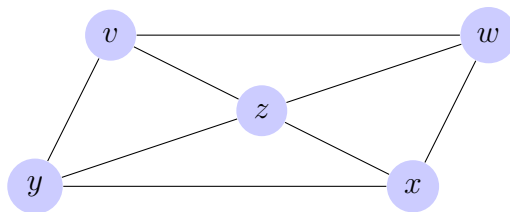
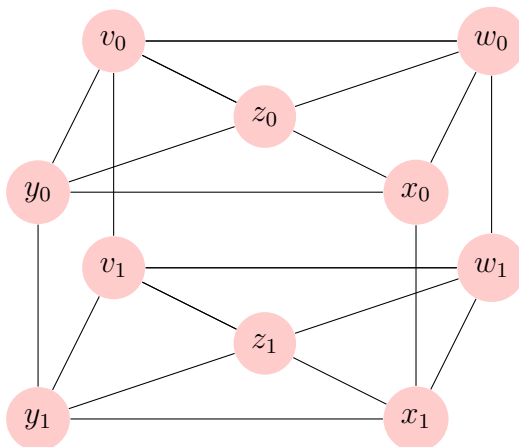
We define $G_{\mathbf{p}}$ to be the random spanning subgraph of G that is obtained from G by the following edge percolation process: for each edge $e \in E(G)$, independently delete e with probability $1 - \mathbf{p}(e)$. Thus $G_{\mathbf{p}}$ takes on values in $\text{span}(G) = \{H : H \text{ is a spanning subgraph of } G\}$. If $p \in [0, 1]$ we write G_p to denote $G_{\mathbf{p}}$ where $\mathbf{p}(e) = p$ for all $e \in E(G)$.

If $x, y \in V(G)$, we write " $x \leftrightarrow y$ in $G_{\mathbf{p}}$ " for the event that $G_{\mathbf{p}}$ contains a path from x to y .

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FIGURE 1.1. A graph G .FIGURE 1.2. The graph $B = BB(G, \{v, w, x, y\})$.

We have the following conjecture.

Conjecture 1.1 (The Bunk Bed Conjecture).

Let G be a graph, $T \subseteq V(G)$, and $B = BB(G, T)$. Let $\mathbf{p} : E(B) \rightarrow [0, 1]$ be symmetric. Then, for all $x, y \in V(G)$, we have

$$P(x_0 \leftrightarrow y_0 \text{ in } B_{\mathbf{p}}) \geq P(x_0 \leftrightarrow y_1 \text{ in } B_{\mathbf{p}}).$$

This conjecture and several special cases have been part of probability folklore since 1985 or earlier; see Section 1, note 5 of (van den Berg and Kahn, 2001) or Conjecture 2.1 of (Linusson, 2011). Although Conjecture 1.1 is intuitively plausible and some special cases have been proved (it is true if G is outerplanar (Linusson, 2011)), no complete proof has been found. We now state two special cases of Conjecture 1.1, each obtained by restricting to constant valued \mathbf{p} .

Conjecture 1.2 (The uniform version of the Bunk Bed Conjecture).

Fix $p \in (0, 1)$. Let G be a graph, $T \subseteq V(G)$, and $B = BB(G, T)$. Then for all $x, y \in V(G)$ we have

$$P(x_0 \leftrightarrow y_0 \text{ in } B_p) \geq P(x_0 \leftrightarrow y_1 \text{ in } B_p).$$

If we take $p = 1/2$ in Conjecture 1.2 we get:

Conjecture 1.3 (The counting version of the Bunk Bed Conjecture).

Let G be a graph, $T \subseteq V(G)$, and $B = BB(G, T)$. Then for all $x, y \in V(G)$ the number of spanning subgraphs of B in which x_0 and y_0 are connected is at least the number of spanning subgraphs in which x_0 and y_1 are connected.

If G is a graph, an orientation of G is an assignment of a direction to each edge of G . Let \vec{G} be an orientation of G chosen uniformly at random from all of the possible orientations of G . If $x, y \in V(G)$ we may then speak of events such as " $x \rightarrow y$ in \vec{G} ", the event that there is a directed path from x to y in \vec{G} . The following theorem of (McDiarmid, 1980) and (Karp, 1990) is of interest.

Theorem 1.4. *Let G be a graph and $x, y \in V(G)$. Then for all $x, y \in V(G)$ we have*

$$P(x \leftrightarrow y \text{ in } G_{1/2}) = P(x \rightarrow y \text{ in } \vec{G}).$$

With this result in hand, the following conjecture is equivalent to Conjecture 1.3.

Conjecture 1.5 (The directed version of the Bunk Bed Conjecture).

Let G be a graph, $T \subseteq V(G)$, and $B = BB(G, T)$. Then for all $x, y \in V(G)$ we have

$$P(x_0 \rightarrow y_0 \text{ in } \vec{B}) \geq P(x_0 \rightarrow y_1 \text{ in } \vec{B}).$$

The main results of this paper are the following theorem and its immediate corollary.

Theorem 1.6. *If $p \in (0, 1)$ is fixed and*

$$P(x_0 \leftrightarrow y_0 \text{ in } BB(G, T)_p) \geq P(x_0 \leftrightarrow y_1 \text{ in } BB(G, T)_p)$$

for all graphs G , all subsets $T \subseteq V(G)$, and all vertices $x, y \in V(G)$, then Conjecture 1.1 is true.

Corollary 1.7. *Conjectures 1.1, 1.2, 1.3, and 1.5 are all equivalent to one another.*

We prove Theorem 1.6 by simulation: we create a sequence of new bunk bed graphs B_n from B by replacing the edges of B with other graphs in such a way that

$$P(u \leftrightarrow v \text{ in } (B_n)_p) \rightarrow P(u \leftrightarrow v \text{ in } B_p) \text{ as } n \rightarrow \infty,$$

where the convergence is uniform over all choices of vertices u, v that are original vertices of B .

We summarize this technique of simulation by edge-replacements in Lemmas 2.1 and 2.2 in Section 2. We then prove Theorem 1.6 in Section 3.

Notes.

(i) Conjecture 1.2 is an often-mentioned variant of Conjecture 1.1, see (Linusson, 2011) for example. The equivalence of the two apparently has never been noted in print. As far as the authors know the statement of Conjecture 1.5 is new. Linusson uses Theorem 1.4 to formulate another conjecture related to Conjecture 1.8 below (Linusson, 2011).

(ii) In Conjecture 1.1, it is equivalent to assume that all posts in $BB(G, T)$ have probability 1. This is because the posts $\{x_0, x_1\}$ of $BB(G, T)$ may be replaced by internally vertex disjoint paths x_0, x'_0, x'_1, x_1 where x'_0 and x'_1 are added vertices, $\{x'_0, x'_1\}$ is a new post of probability 1 and edges $\{x_0, x'_0\}$ and $\{x'_1, x_1\}$ of the new upper and lower bunks respectively are each given probability \sqrt{p} . This may also be seen by applying the law of total probability, conditioning on the outcomes of the random variable $E(B_p) \cap E_T(B)$. In Conjectures 1.2 and 1.3 it is likewise equivalent to assume that posts are always present.

(iii) One may also define $BB_i(G, T)$, the bunk bed graph consisting of G_0 and G_1 with x_0 and x_1 identified for each $x \in T$ instead of connected by a post. Note that $BB_i(G, T)$ may be a multi-graph; an edge $e = \{x, y\} \subseteq T$ will become two distinct edges e_0 and e_1 in $BB_i(G, T)$. Since posts may assumed to be always present, the statements of Conjectures 1.1, 1.2, 1.3, and 1.5 with $BB(G, T)$ replaced by $BB_i(G, T)$ are also all equivalent to each other and Conjecture 1.1. In fact, Conjecture 1.1 is stated in this form in (van den Berg and Kahn, 2001).

(iv) The following special case of Conjecture 1.1 also appears in the literature: as Question 3.1 of (Häggström, 1998), Conjecture 2.1 of (Häggström, 2003), and Conjecture 1.1 of (Linusson, 2011).

Conjecture 1.8. Fix $p \in (0, 1)$. Let G be a graph and $B = BB(G, V(G))$. Then, for all $x, y \in V(G)$, we have

$$P(x_0 \leftrightarrow y_0 \text{ in } B_p) \geq P(x_0 \leftrightarrow y_1 \text{ in } B_p).$$

The case where $G = K_n$ and $p = 1/2$ has recently been proven (de Buyer, 2016). We have not been able to show that Conjecture 1.8 is equivalent to the Bunk Bed Conjecture. Perhaps it is not.

2. Simulation

We now formalize the process of simulation in Lemmas 2.1 and 2.2. For notational convenience we will denote $G_{\mathbf{p}}$ by $G(\mathbf{p})$ in the following exposition.

Let G be a graph. For each $e \in E(G)$, let $(H_e, \mathbf{p}_e, x_e, y_e)$ be a 4-tuple where H_e is a graph, $\mathbf{p}_e : E(H_e) \rightarrow [0, 1]$ is an edge-probability function on H_e , and x_e and y_e are distinct vertices of H_e . Let G' be the graph obtained from the vertex disjoint union $G'' = V(G) \sqcup \bigsqcup_{e \in E(G)} H_e$ by successively identifying vertices x and x_e and y and y_e for each edge $e = \{x, y\} \in E(G)$. The vertex resulting from the identifications of $x \in V(G)$ with other vertices in G'' will also be called x and we view $V(G)$ as a subset of $V(G')$. Likewise, each edge $f \in E(G')$ is viewed as belonging to the unique H_e from which it arose in G' . Let $\mathbf{p}' : E(G') \rightarrow [0, 1]$ be defined by setting $\mathbf{p}'(f) = \mathbf{p}_e(f)$ if and only if $f \in E(H_e)$. Informally, G' is the graph obtained from G by replacing each edge e with H_e and \mathbf{p}' is the edge-probability function on G' that restricts to \mathbf{p}_e on H_e .

We define a map $\pi : \text{span}(G') \rightarrow \text{span}(G)$ as follows. If $H' \in \text{span}(G')$ then $H = \pi(H')$ is the spanning subgraph of G that retains $e = \{x_e, y_e\} \in E(G)$ if and only if $x_e \leftrightarrow y_e$ in $H' \cap H_e$. We define $\mathbf{q} : E(G) \rightarrow [0, 1]$ by setting $\mathbf{q}(e) = P(x_e \leftrightarrow y_e \text{ in } H_e(\mathbf{p}_e))$ for $e \in E(G)$.

Lemma 2.1. *The following statements hold.*

- (i) *If $H \in \text{span}(G)$, then $P(G(\mathbf{q}) = H) = P(G'(\mathbf{p}') \in \pi^{-1}(H))$.*
- (ii) *If $H' \in \text{span}(G')$ and $H = \pi(H')$, then, for all $x, y \in V(G)$, $x \leftrightarrow y$ in H' if and only if $x \leftrightarrow y$ in H .*
- (iii) *For all $x, y \in V(G)$, $P(x \leftrightarrow y \text{ in } G(\mathbf{q})) = P(x \leftrightarrow y \text{ in } G'(\mathbf{p}'))$.*

We omit the proof of Lemma 2.1; the details are straightforward but tedious to verify.

Lemma 2.2. *Let G be a graph. For all $\epsilon > 0$ there is a $\gamma > 0$ such that for all $\mathbf{p}, \mathbf{q} : E(G) \rightarrow [0, 1]$, if $|\mathbf{p}(e) - \mathbf{q}(e)| < \gamma$ for all $e \in E(G)$ then*

$$|P(G_{\mathbf{p}} \in A) - P(G_{\mathbf{q}} \in A)| < \epsilon$$

for all $A \subseteq \text{span}(G)$.

Proof. For each $A \subseteq \text{span}(G)$, $f_A(\mathbf{p}) = P(G_{\mathbf{p}} \in A)$ is a polynomial in the variables $\mathbf{p}(e)$. Thus f_A is continuous and even uniformly continuous for \mathbf{p} in the compact set $[0, 1]^{E(G)}$. Since G is finite, there are only finitely many $A \subseteq \text{span}(G)$, so given $\epsilon > 0$ we can find a bound $\gamma > 0$ that will apply uniformly to all f_A . \square

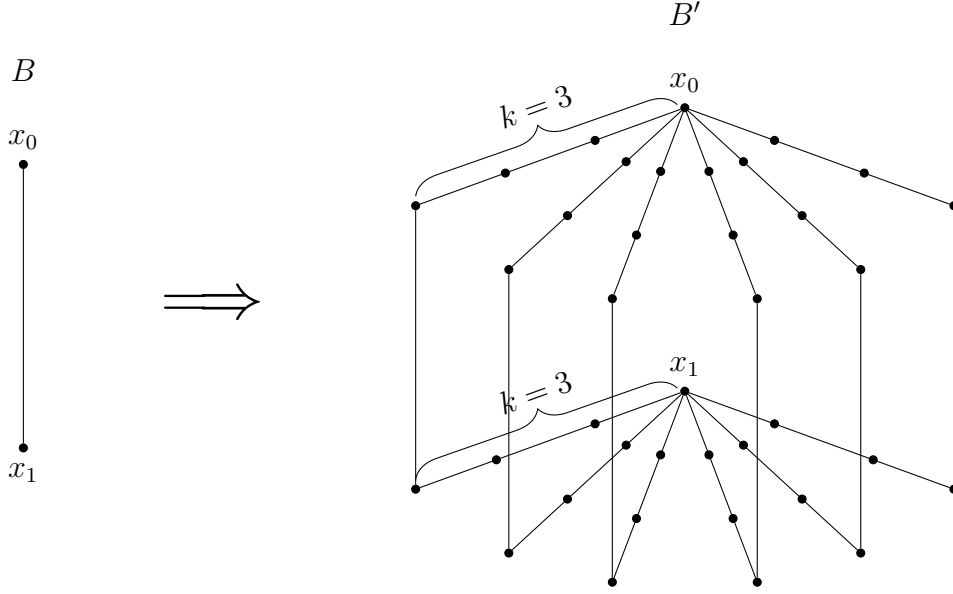


FIGURE 3.1. Post $\{x_0, x_1\}$ in B is replaced by $F_{6,7}$ and 6 posts in B' .

3. Proof of Theorem 1.6

We use the following class of 4-tuples as replacements in our proof of Theorem 1.6. For integers $j, k \geq 1$ and $p \in (0, 1)$ we define $F_{j,k}(p) = (H_{j,k}, \mathbf{p}, h_0, h_k)$ where $H_{j,k}$ is the graph consisting of vertices h_0 and h_k and j internally vertex disjoint paths from h_0 to h_k , each of length k , and where $\mathbf{p}(e) = p$ for all $e \in E(H_{j,k})$. Theorem 1.6 is an immediate consequence of Lemma 3.1, listed below.

Lemma 3.1. *Let G be a graph, $T \subseteq V(G)$, and $B = BB(G, T)$. Let $\mathbf{p} : E(B) \rightarrow [0, 1]$ be symmetric. Let $p \in (0, 1)$ and $\epsilon > 0$. Then there is another bunk bed graph $B' = BB(G', T')$ with $V(G) \subseteq V(G')$ and $T' \subseteq V(G')$, such that for all $x, y \in V(G)$ and $i, j \in \{0, 1\}$,*

$$|P(x_i \leftrightarrow y_j \text{ in } B_{\mathbf{p}}) - P(x_i \leftrightarrow y_j \text{ in } B'_p)| < \epsilon. \quad (3.1)$$

Proof. Fix $p \in (0, 1)$ and $\epsilon > 0$. Let G be a graph, $T \subseteq V(G)$, and $B = BB(G, T)$. Let $\mathbf{p} : E(B) \rightarrow [0, 1]$ be symmetric. We construct B' as follows.

For each edge $e \in E(G)$, we pick $j = j_e \geq 1$ and $k = k_e \geq 1$, and replace $e_0 \in E(B)$ by a copy of $F_{j,k}(p)$ in the upper bunk of B' and $e_1 \in E(B)$ by another copy of $F_{j,k}(p)$ in the lower bunk of B' . For each edge $e = \{x_0, x_1\} \in E(B)$ with $x \in T$ we pick $j = j_e \geq 1$ and $k = k_e \geq 1$ and replace e by $F_{j,2k+1}(p)$. This action adds j x_0 - x_1 paths to B , each of length $2k + 1$. For each such path $P = (x_0, h_1, \dots, h_k, h_{k+1}, \dots, h_{2k}, x_1)$, we view the sub-path $x_0 P h_k$ as belonging to the upper bunk of B' , $h_{k+1} P x_1$ as belonging to the lower bunk of B' , and the edge $\{h_k, h_{k+1}\}$ as being a post of B' . These new posts are the post edges of B' . See Figure 3.1.

Let $\mathbf{p}' : E(B') \rightarrow [0, 1]$ and $\mathbf{q} : E(B) \rightarrow [0, 1]$ be as in Lemma 2.1. Note that $\mathbf{p}'(e) = p$ for all $e \in E(B')$. Note that \mathbf{q} is symmetric. By Lemma 2.2, we can pick $\gamma > 0$ so that if

$$|\mathbf{q}(e) - \mathbf{p}(e)| < \gamma \text{ for all } e \in E(B) \quad (3.2)$$

then (3.1) will hold. It is not hard to show that we can pick j_e and k_e so that (3.2) will hold. We can take each k_e to be odd and then each post edge e has $q(e) = 1 - (1 - p^{2k+1})^j$ for some $j, k \geq 1$. These numbers are dense in $[0, 1]$. □

Proof. (of Theorem 1.6) Let G, T, B , and \mathbf{p} be as in the statement of Conjecture 1.1. Fix $p \in (0, 1)$. If we had

$$\delta = P(x_0 \leftrightarrow y_1 \text{ in } B_{\mathbf{p}}) - P(x_0 \leftrightarrow y_0 \text{ in } B_{\mathbf{p}}) > 0$$

for some $x, y \in V(G)$, then by applying Lemma 3.1 with $\epsilon = \delta/3$, we would have

$$P(x_0 \leftrightarrow y_1 \text{ in } B'_p) - P(x_0 \leftrightarrow y_0 \text{ in } B'_p) > \delta/3 > 0,$$

contradicting the assumption of the theorem. □

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