North Carolina Journal of
Mathematics and Statistics
Volume 3, Pages 7–20 (Accepted June 23, 2017, published June 30, 2017)
ISSN 2380-7539

A Mathematical Analysis on the Transmission Dynamics of Neisseria gonorrhoeae

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ABSTRACT. In this project, we analyze an epidemiological model describing the transmission of gonorrhea. We address two stratifications: one based on age groups and one based on education levels, each with a core sexual activity class and two noncore sexual activity classes. Using parameters based on sexual behavior in the United States, we address the impact of the average number of partners per year for each sexual activity class on the behavior of the model around two equilibrium points: a disease-free equilibrium and an endemic equilibrium. The focus of the project is to identify the conditions leading to the existence of each of the equilibrium points, analyze the local stability of these points, and discuss the results. Ultimately, the goal of the project is to find conditions for the bifurcation of the two equilibrium points, in order to obtain the highest average number of partners per year for various groups resulting in the eradication of gonorrhea.

1. Introduction

The application of compartmental models to epidemiology began with the SIR Model, developed by Kermack and McKendrick (Brauer, 2008). Over the past century, this model has been analyzed and expanded to fit the dynamics of various diseases. The model discussed in this paper was presented by Garnett and Anderson (1996) and employs compartments with terms relevant to the dynamics of gonorrhea.

The model that Garnett and Anderson (1996) present is as follows:

$$\begin{aligned} \frac{dX_{ki}}{dt} &= \mu N_{ki} - \beta_k c_{ki} X_{ki} \sum_{j=1}^{n} \rho_{kij} \left(\frac{Y_{k'j}}{N_{k'j}} \right) - \mu X_{ki} + \nu Y_{ki}, \\ \frac{dY_{ki}}{dt} &= \beta_k c_{ki} X_{ki} \sum_{j=1}^{n} \rho_{kij} \left(\frac{Y_{k'j}}{N_{k'j}} \right) - (\nu + \mu) Y_{ki}, \end{aligned} \tag{1.1}$$

where X_{ki} is the susceptible population, Y_{ki} is the infectious population, and N_{ki} is the total population. Subscript k represents sex (male or female), with k' as the opposite sex, and subscript i represents the sexual activity class. Parameter μ is the rate of both entry to and exit from the sexually active population, β_k is the transmission probability per partnership from sex k' to sex k, c_{ki} is the rate of sex partner change of sex k in activity group i, ρ_{kij} defines the probability that someone of sex k in activity group i mates with someone of sex k' in activity group j, and ν is the recovery rate.

Received by the editors March 31, 2017. Revised version received June 23, 2017.
2010 *Mathematics Subject Classification.* 37N25; 92D30; 34D20.
Key words and phrases. Infectious disease; Mathematical modeling; Ordinary differential equations; Stability analysis; Epidemic control.

The term μN_{ki} represents individuals becoming sexually active and therefore becoming susceptible to gonorrhea. The term $\beta_k c_{ki} X_{ki} \sum_{j=1}^{n} \rho_{kij} \left(\frac{Y_{k'j}}{N_{k'j}}\right)$ is the rate individuals become infected. The terms μX_{ki} and μY_{ki} are the rates individuals become sexually inactive while susceptible or infectious, respectively, and are no longer transmitting gonorrhea. The term νY_{ki} represents the rate at which individuals recover. These dynamics are displayed visually in Figure 1.1.

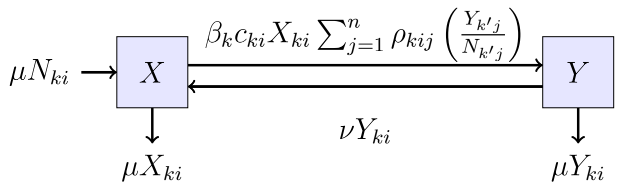

FIGURE 1.1. Garnett and Anderson Transmission Dynamics.

We assume that the average number of partners per year and the mixing probabilities are equivalent for both sexes, thereby eliminating subscript k (Garnett and Anderson, 1996). Furthermore, we assume the total population is constant and $N_{ki} = X_{ki} + Y_{ki}$. Since $X_i = N_i - Y_i$, System (1.1) can be reduced to a system of ordinary differential equations consisting solely of the Y_i functions.

Our analysis considers three sexual activity groups, a core group and two noncore groups, with parameter values specific for the disease and to human activity. Research on gonorrhea has found that the average length of infection is six months, or $\nu = 2.0$ recoveries per year (Garnett and Anderson, 1993). The average transmission probability per partnership is eighty percent, or $\beta = 0.8$ (Garnett and Anderson, 1993). Also, research on sexual activity lifespan has found the average person to be sexually active for forty years, or $\mu = 0.025$ years^{-1} (Lindau and Gavrilova, 2010). Therefore, the system becomes:

$$\begin{aligned}
Y_1' &= 0.8c_1 (N_1 - Y_1) \left[\rho_{11}\left(\frac{Y_1}{N_1}\right) + \rho_{12}\left(\frac{Y_2}{N_2}\right) + \rho_{13}\left(\frac{Y_3}{N_3}\right)\right] - 2.025Y_1, \\
Y_2' &= 0.8c_2 (N_2 - Y_2) \left[\rho_{21}\left(\frac{Y_1}{N_1}\right) + \rho_{22}\left(\frac{Y_2}{N_2}\right) + \rho_{23}\left(\frac{Y_3}{N_3}\right)\right] - 2.025Y_2, \\
Y_3' &= 0.8c_3 (N_3 - Y_3) \left[\rho_{31}\left(\frac{Y_1}{N_1}\right) + \rho_{32}\left(\frac{Y_2}{N_2}\right) + \rho_{33}\left(\frac{Y_3}{N_3}\right)\right] - 2.025Y_3,
\end{aligned} \tag{1.2}$$

where $Y_i' = \frac{dY_i}{dt}$, $i = 1$ is the core group, and $i = 2, 3$ are the noncore groups.

The purpose of this paper is to determine the highest average number of partners per year for different groups of individuals that ensure the asymptotic eradication of gonorrhea. In Section 2, we consider a stratification based on age. We describe the process of calculating parameters specific to this stratification and provide two examples of the analysis process, complete with numerical simulations. By varying the group contact rates, we determine the highest average number of partners per year for individuals less than nineteen years of age, between the ages of twenty and twenty-nine, and over the age of thirty such that the disease-free equilibrium, or where $(Y_1, Y_2, Y_3) = (0, 0, 0)$, is locally asymptotically stable. Similarly, in Section 3, we consider a stratification based on education level. Once again, we vary the group contact rates to determine the highest average number of partners per year for individuals with less than a high school education, those with a high school diploma or GED, and those with more than a high school education such that the disease-free equilibrium is locally asymptotically stable. The results from each stratification may be integrated by public health departments into educational materials and other preventative methods.

2. Age stratification

In this section, we consider a stratification of the heterosexual, sexually active population based on age. According to research by Aral et al. (1999), the age group with the highest prevalence of gonorrhea are those less than nineteen years old. Then, $i = 1$ are individuals less than nineteen years old, $i = 2$ are individuals between the ages of twenty and twenty-nine, and $i = 3$ are individuals over thirty years old.

2.1. Parameterization

We calculate values for N_i and ρ_{ij} based on data collected by the CDC and the research of Aral et al.. The CDC's 2011 National Health Statistics Report (NHSR) details the number of individuals in the United States of each age group, as well as the percentage of those who have either never had sex, or are not currently sexually active (Chandra et al.). From this data, we are able to calculate the number of sexually active individuals in each of the NHSR's age groups by subtracting the sum of the percentages of those who have never had heterosexual sexual contact and those who have not had contact in the last year (and are therefore not currently sexually active) in each age group from the total population of that age group. These values are shown in Table 2.1.

TABLE 2.1. Number of Sexually Active Women and Men of NHSR Age Groups.

	Age	Sexually Active Women	Sexually Active Men
k		n_{wk}	n_{mk}
1	15-19 years	4933863	5410054
2	20-24 years	8416200	8302392
3	25-29 years	9327500	9450486
4	30-34 years	8963845	8818575
5	35-39 years	9678900	9595740
6	40-44 years	9949692	9593415

The total heterosexual, sexually active populations in each of the sexual activity groups are then:

$$
\begin{aligned}
N_1 &= n_{w1} + n_{m1} = 10343917, \\
N_2 &= n_{w2} + n_{w3} + n_{m2} + n_{m3} = 35496578, \\
N_3 &= n_{w4} + n_{w5} + n_{w6} + n_{m4} + n_{m5} + n_{m6} = 56600167.
\end{aligned} \tag{2.1}
$$

We then determined the values in the mixing matrix, or the matrix defining the likelihood that two individuals of each sexual activity group will mate. Using the data published for each sex and age group by Aral et al. and weighting these values with the corresponding adjusted populations from the NHSR, the mixing matrix for the age stratification becomes:

$$\begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{bmatrix} = \begin{bmatrix} 0.665 & 0.324 & 0.010 \\ 0.143 & 0.684 & 0.173 \\ 0.024 & 0.352 & 0.624 \end{bmatrix} \tag{2.2}$$

Substituting the values from (2.1) and (2.2) into System (1.2), we have

$$\begin{aligned} Y_1' &= \beta c_1 (N_1 - Y_1) \left[\rho_{11} \left(\frac{Y_1}{N_1} \right) + \rho_{12} \left(\frac{Y_2}{N_2} \right) + \rho_{13} \left(\frac{Y_3}{N_3} \right) \right] - (\nu + \mu) Y_1 \\ &= 0.8 c_1 (10343917 - Y_1) \\ &\quad \times \left(6.42989 \times 10^{-8} Y_1 + 9.13124 \times 10^{-9} Y_2 + 1.81038 \times 10^{-10} Y_3 \right) - 2.025 Y_1 \\ &= f(Y_1, Y_2, Y_3), \\ Y_2' &= \beta c_2 (N_2 - Y_2) \left[\rho_{21} \left(\frac{Y_1}{N_1} \right) + \rho_{22} \left(\frac{Y_2}{N_2} \right) + \rho_{23} \left(\frac{Y_3}{N_3} \right) \right] - (\nu + \mu) Y_2 \\ &= 0.8 c_2 (35496578 - Y_2) \\ &\quad \times \left(1.37783 \times 10^{-8} Y_1 + 1.92692 \times 10^{-8} Y_2 + 3.06515 \times 10^{-9} Y_3 \right) - 2.025 Y_2 \\ &= g(Y_1, Y_2, Y_3), \\ Y_3' &= \beta c_3 (N_3 - Y_3) \left[\rho_{31} \left(\frac{Y_1}{N_1} \right) + \rho_{32} \left(\frac{Y_2}{N_2} \right) + \rho_{33} \left(\frac{Y_3}{N_3} \right) \right] - (\nu + \mu) Y_3 \\ &= 0.8 c_3 (56600167 - Y_3) \\ &\quad \times \left(2.31521 \times 10^{-9} Y_1 + 9.90420 \times 10^{-9} Y_2 + 1.10331 \times 10^{-8} Y_3 \right) - 2.025 Y_3 \\ &= h(Y_1, Y_2, Y_3). \end{aligned} \tag{2.3}$$

2.2. Analysis

We consider values of c_1, c_2, and c_3, under the condition that $c_1 \geq c_2 \geq c_3 \geq 1.0$ (Chandra et al.). Varying c_1, c_2, and c_3 according to the method described in Appendix A, each set of (c_1, c_2, c_3) is substituted into System (2.3). The partial derivatives of f, g, and h with respect to Y_1, Y_2, and Y_3 are taken to form the Jacobian matrix. The equilibrium points of the system are determined by setting $f = g = h = 0$ and solving for Y_1, Y_2, and Y_3. For each equilibrium point, the values for Y_1, Y_2, and Y_3 are substituted into the Jacobian matrix; the eigenvalues of the Jacobian matrix are obtained and analyzed to determine the local stability of the system.

2.2.1. Example 1

For example, we consider the point $(c_1, c_2, c_3) = (2.7, 2.5, 2.4)$. Substituting these c_i values into System (2.3), the system becomes

$$\begin{aligned}
Y_1' &= 2.16\,(10343917 - Y_1) \\
&\quad \times \left(6.42989 \times 10^{-8} Y_1 + 9.13124 \times 10^{-9} Y_2 + 1.81038 \times 10^{-10} Y_3\right) - 2.025 Y_1 \\
&= f_1\,(Y_1, Y_2, Y_3)\,, \\
Y_2' &= 2.00\,(35496578 - Y_2) \\
&\quad \times \left(1.37783 \times 10^{-8} Y_1 + 1.92692 \times 10^{-8} Y_2 + 3.06515 \times 10^{-9} Y_3\right) - 2.025 Y_2 \\
&= g_1\,(Y_1, Y_2, Y_3)\,, \\
Y_3' &= 1.92\,(56600167 - Y_3) \\
&\quad \times \left(2.31521 \times 10^{-9} Y_1 + 9.90420 \times 10^{-9} Y_2 + 1.10331 \times 10^{-8} Y_3\right) - 2.025 Y_3 \\
&= h_1\,(Y_1, Y_2, Y_3)\,.
\end{aligned} \tag{2.4}$$

The partial derivatives of f_1, g_1, and h_1 with respect to Y_1, Y_2, and Y_3 are taken to form the Jacobian matrix J_1 for System (2.4):

$$J_1 = \begin{bmatrix} \frac{\partial f_1}{\partial Y_1} & \frac{\partial f_1}{\partial Y_2} & \frac{\partial f_1}{\partial Y_3} \\ \frac{\partial g_1}{\partial Y_1} & \frac{\partial g_1}{\partial Y_2} & \frac{\partial g_1}{\partial Y_3} \\ \frac{\partial h_1}{\partial Y_1} & \frac{\partial h_1}{\partial Y_2} & \frac{\partial h_1}{\partial Y_3} \end{bmatrix}, \tag{2.5}$$

where

$$\begin{aligned}
\frac{\partial f_1}{\partial Y_1} &= -2.77771 \times 10^{-7} Y_1 - 1.97235 \times 10^{-8} Y_2 - 3.91043 \times 10^{10} Y_3 - 0.58838, \\
\frac{\partial f_1}{\partial Y_2} &= 0.20402 - 1.97235 \times 10^{-8} Y_1, \\
\frac{\partial f_1}{\partial Y_3} &= 0.00404 - 3.91043 \times 10^{-10} Y_1, \\
\frac{\partial g_1}{\partial Y_1} &= 0.97817 - 2.75566 \times 10^{-8} Y_2, \\
\frac{\partial g_1}{\partial Y_2} &= -2.75566 \times 10^{-8} Y_1 - 7.70767 \times 10^{-8} Y_2 - 6.13031 \times 10^{-9} Y_3 - 0.65702, \\
\frac{\partial g_1}{\partial Y_3} &= 0.21760 - 6.13031 \times 10^{-9} Y_2, \\
\frac{\partial h_1}{\partial Y_1} &= 0.25160 - 4.44520 \times 10^{-9} Y_3, \\
\frac{\partial h_1}{\partial Y_2} &= 1.07631 - 1.90161 \times 10^{-8} Y_3, \\
\frac{\partial h_1}{\partial Y_3} &= -4.44520 \times 10^{-9} Y_1 - 1.90161 \times 10^{-8} Y_2 - 4.23672 \times 10^{-8} Y_3 - 0.82601.
\end{aligned}$$

Setting $f_1 = g_1 = h_1 = 0$ and solving for Y_1, Y_2, and Y_3, one equilibrium point exists: $(Y_1, Y_2, Y_3) = (0, 0, 0)$, which is the disease-free equilibrium.

The equilibrium values $(Y_1, Y_2, Y_3) = (0, 0, 0)$ are substituted into Matrix (2.5):

$$J_1(0,0,0) = \begin{bmatrix} -0.58838 & 0.20402 & 0.00404 \\ 0.97817 & -0.65702 & 0.21760 \\ 0.25160 & 1.07631 & -0.82601 \end{bmatrix} \tag{2.6}$$

The eigenvalues of $J_1(0,0,0)$ are found to be $\lambda_1 = -0.00058$, $\lambda_2 = -0.73079$, and $\lambda_3 = -1.34003$. Since $\lambda_1, \lambda_2, \lambda_3 < 0$, then the equilibrium $(Y_1, Y_2, Y_3) = (0, 0, 0)$, the disease-free equilibrium, is locally asymptotically stable at $(c_1, c_2, c_3) = (2.7, 2.5, 2.4)$. This asymptotic behavior is depicted in Figure 2.1, with initial conditions chosen to be $Y_1(0) = 70000$, $Y_2(0) = 200000$, and $Y_3(0) = 100000$, based on data from CDC et al. (2015).

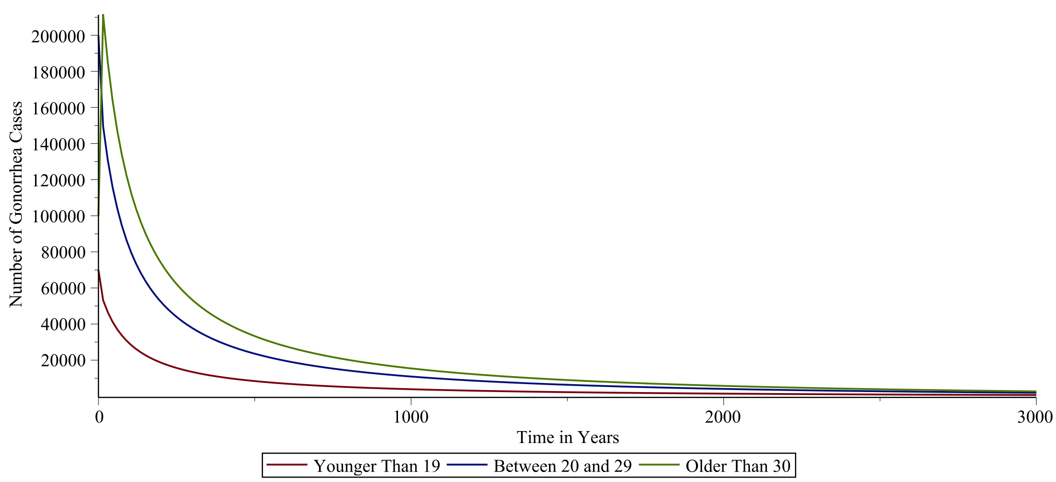

FIGURE 2.1. Asymptotic Behavior of Age Stratification with $(c_1, c_2, c_3) = (2.7, 2.5, 2.4)$.

2.2.2. Example 2

Continuing to vary c_i, we consider the point $(c_1, c_2, c_3) = (3.2, 2.2, 2.0)$. Substituting these c_i values into System (2.3), the system becomes

$$\begin{aligned} Y_1' &= 2.56\,(10343917 - Y_1) \\ &\quad \times \left(6.42989 \times 10^{-8} Y_1 + 9.13124 \times 10^{-9} Y_2 + 1.81038 \times 10^{-10} Y_3\right) - 2.025 Y_1 \\ &= f_2\,(Y_1, Y_2, Y_3)\,, \\ Y_2' &= 1.76\,(35496578 - Y_2) \\ &\quad \times \left(1.37783 \times 10^{-8} Y_1 + 1.92692 \times 10^{-8} Y_2 + 3.06515 \times 10^{-9} Y_3\right) - 2.025 Y_2 \\ &= g_2\,(Y_1, Y_2, Y_3)\,, \\ Y_3' &= 1.60\,(56600167 - Y_3) \\ &\quad \times \left(2.31521 \times 10^{-9} Y_1 + 9.90420 \times 10^{-9} Y_2 + 1.10331 \times 10^{-8} Y_3\right) - 2.025 Y_3 \\ &= h_2\,(Y_1, Y_2, Y_3)\,. \end{aligned} \tag{2.7}$$

The partial derivatives of f_2, g_2, and h_2 with respect to Y_1, Y_2, and Y_3 are taken to form the Jacobian matrix J_2 for System (2.7):

$$J_2 = \begin{bmatrix} \frac{\partial f_2}{\partial Y_1} & \frac{\partial f_2}{\partial Y_2} & \frac{\partial f_2}{\partial Y_3} \\ \frac{\partial g_2}{\partial Y_1} & \frac{\partial g_2}{\partial Y_2} & \frac{\partial g_2}{\partial Y_3} \\ \frac{\partial h_2}{\partial Y_1} & \frac{\partial h_2}{\partial Y_2} & \frac{\partial h_2}{\partial Y_3} \end{bmatrix}, \tag{2.8}$$

where

$$\begin{aligned} \frac{\partial f_2}{\partial Y_1} &= -3.29210 \times 10^{-7} Y_1 - 2.33760 \times 10^{-8} Y_2 - 4.63458 \times 10^{-10} Y_3 - 0.32234, \\ \frac{\partial f_2}{\partial Y_2} &= 0.24180 - 2.33760 \times 10^{-8} Y_1, \\ \frac{\partial f_2}{\partial Y_3} &= 0.00479 - 4.63458 \times 10^{-10} Y_1, \\ \frac{\partial g_2}{\partial Y_1} &= 0.86079 - 2.42499 \times 10^{-8} Y_2, \\ \frac{\partial g_2}{\partial Y_2} &= -2.42499 \times 10^{-8} Y_1 - 6.78275 \times 10^{-8} Y_2 - 5.39467 \times 10^{-9} Y_3 - 0.82118, \\ \frac{\partial g_2}{\partial Y_3} &= 0.19149 - 5.39467 \times 10^{-9} Y_2, \\ \frac{\partial h_2}{\partial Y_1} &= 0.20967 - 3.70434 \times 10^{-9} Y_3, \\ \frac{\partial h_2}{\partial Y_2} &= 0.89693 - 1.58467 \times 10^{-8} Y_3, \\ \frac{\partial h_2}{\partial Y_3} &= -3.70434 \times 10^{-9} Y_1 - 1.58467 \times 10^{-8} Y_2 - 3.53060 \times 10^{-}8 Y_3 - 1.02584. \end{aligned}$$

Setting $f_2 = g_2 = h_2 = 0$ and solving for Y_1, Y_2, and Y_3, two equilibrium points exist: $(Y_1, Y_2, Y_3) = (0, 0, 0)$, which is the disease-free equilibrium, and at $(Y_1, Y_2, Y_3) = (Y_1^*, Y_2^*, Y_3^*) = (71231.90049, 97128.91127, 99138.62909)$, which is the endemic equilibrium.

Considering the disease-free equilibrium, $(Y_1, Y_2, Y_3) = (0, 0, 0)$ are substituted into Matrix (2.8):

$$J_2(0,0,0) = \begin{bmatrix} -0.32234 & 0.24180 & 0.00479 \\ 0.86079 & -0.82118 & 0.19149 \\ 0.20967 & 0.89693 & -1.02584 \end{bmatrix}. \tag{2.9}$$

The eigenvalues of $J_2(0,0,0)$ are found to be $\lambda_1 = 0.01072$, $\lambda_2 = -0.76148$, and $\lambda_3 = -1.41860$. Since $\lambda_1 > 0$, then the equilibrium $(Y_1, Y_2, Y_3) = (0, 0, 0)$, the disease-free equilibrium, is unstable at $(c_1, c_2, c_3) = (3.2, 2.2, 2.0)$.

Considering the endemic equilibrium, $(Y_1, Y_2, Y_3) = (Y_1^*, Y_2^*, Y_3^*)$ are substituted into Matrix (2.8):

$$J_2(Y_1^*, Y_2^*, Y_3^*) = \begin{bmatrix} -0.34810 & 0.24013 & 0.00476 \\ 0.85843 & -0.83003 & 0.19097 \\ 0.20930 & 0.89536 & -1.03114 \end{bmatrix}. \tag{2.10}$$

The eigenvalues of $J_2(Y_1^*, Y_2^*, Y_3^*)$ are found to be $\lambda_1 = -0.01068$, $\lambda_2 = -0.77326$, and $\lambda_3 = -1.42533$. Since $\lambda_1, \lambda_2, \lambda_3 < 0$, then the equilibrium $(Y_1, Y_2, Y_3) = (Y_1^*, Y_2^*, Y_3^*)$, the endemic equilibrium, is locally asymptotically stable at $(c_1, c_2, c_3) = (3.2, 2.2, 2.0)$. This asymptotic behavior is depicted in Figure (2.2), with initial conditions chosen to be $Y_1(0) = 70000$, $Y_2(0) = 200000$, and $Y_3(0) = 100000$, based on CDC et al. (2015).

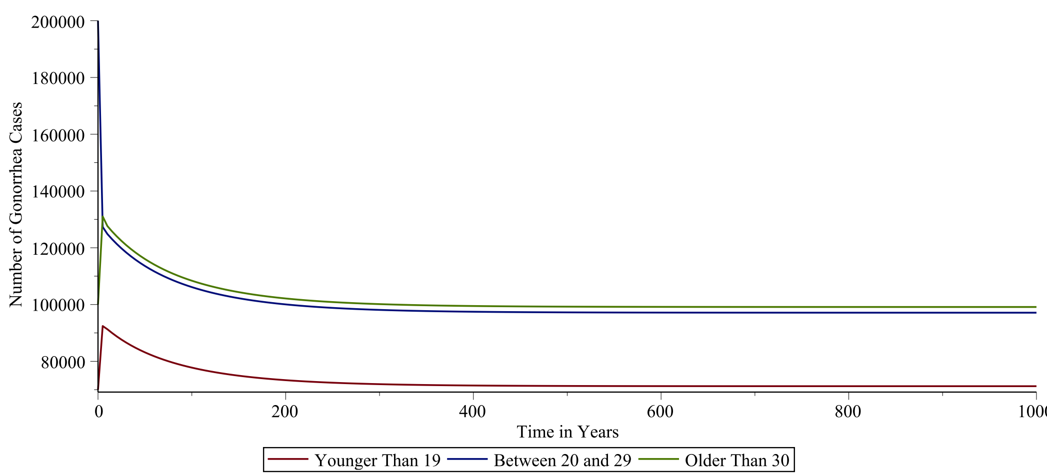

FIGURE 2.2. Asymptotic Behavior of Age Stratification with $(c_1, c_2, c_3) = (3.2, 2.2, 2.0)$.

2.3. Results

Throughout this analysis, a bifurcation emerged. The points in Table 2.2 represent the largest values of c_i at 0.1 increments such that the disease-free equilibrium is locally asymptotically stable, where the eigenvalues of the Jacobian matrix are all with negative real parts. From these critical points with three group contact rates, if at least one contact rate is raised, then the endemic equilibrium becomes locally asymptotically stable, which predicts the persistence of the disease. The method in Appendix A was then refined to 0.01 increments in Python, with the resulting c_i values plotted in Figure (2.3), along with the surface of best fit. Visually, any (c_1, c_2, c_3) less than or equal to the surface in Figure (2.3) predicts the asymptotic eradication of gonorrhea.

TABLE 2.2. Eradication Points for Age Stratification.

c_1	c_2	c_3	c_1	c_2	c_3	c_1	c_2	c_3	c_1	c_2	c_3	c_1	c_2	c_3
3.6	1.2	1.2	3.2	2.3	1.1	3.0	2.5	1.5	2.8	2.7	1.4	2.7	2.5	2.4
3.5	1.6	1.6	3.2	2.2	1.8	3.0	2.4	1.9	2.8	2.6	1.8	2.6	2.6	2.2
3.4	1.9	1.3	3.2	2.1	2.1	3.0	2.3	2.3	2.8	2.5	2.2	2.6	2.5	2.5
3.4	1.8	1.8	3.1	2.4	1.4	2.9	2.6	1.5	2.8	2.4	2.4			
3.3	2.1	1.4	3.1	2.3	1.9	2.9	2.5	1.9	2.7	2.7	1.7			
3.3	2.0	2.0	3.1	2.2	2.2	2.9	2.4	2.3	2.7	2.6	2.1			

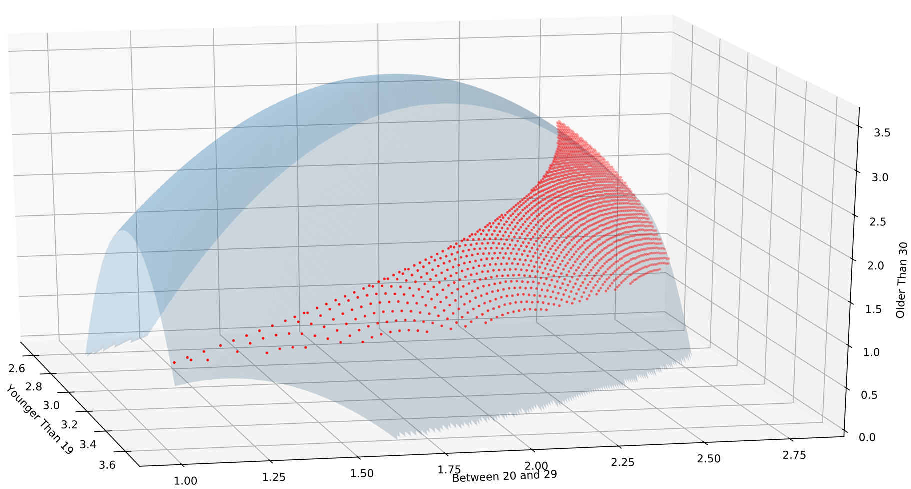

FIGURE 2.3. Approximate Bifurcation Points for Average Partners per Year in Age Stratification with 0.01 Increments.

3. Education Stratification

In this section, we consider a stratification of the heterosexual, sexually active population based on education level. According to research by Aral et al. (1999), the education level with the highest prevalence of gonorrhea are those with less than a high school education. Then, $i = 1$ are individuals with less than a high school education, $i = 2$ are individuals with a high school education, and $i = 3$ are individuals with more than a high school education.

3.1. Parameterization

As in the age stratification, we find values for N_i and ρ_{ij} based on the data in the NHSR and the research of Aral et al. (Chandra et al.; Aral et al., 1999). Using the same method as in the previous stratification, the number of sexually active individuals in each of the NHSR's education levels are shown in Table 3.1.

TABLE 3.1. Number of Sexually Active Women and Men of NHSR Education Levels.

	Education Level	Sexually Active Women	Sexually Active Men
k		n_{wk}	n_{mk}
1	No high school diploma or GED	5762880	7881720
2	High school diploma or GED	10920318	11111590
3	Some college, no bachelor's degree	11939634	11668623
4	Bachelor's degree or higher	13988700	11709016

The total heterosexual, sexually active populations in each of the sexual activity groups are then:

$$\begin{aligned} N_1 &= n_{w1} + n_{m1} = 13644600, \\ N_2 &= n_{w2} + n_{m2} = 22031908, \\ N_3 &= n_{w3} + n_{w4} + n_{m3} + n_{m4} = 49305973. \end{aligned} \tag{3.1}$$

Using the data published for each sex and education group by Aral et al. and weighting these values with the adjusted populations from the NHSR, the mixing matrix for the education stratification becomes:

$$\begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{bmatrix} = \begin{bmatrix} 0.371 & 0.417 & 0.213 \\ 0.187 & 0.532 & 0.281 \\ 0.074 & 0.352 & 0.575 \end{bmatrix} \tag{3.2}$$

Substituting the values from (3.1) and (3.2) into System (1.2), we have

$$\begin{aligned}
Y_1' &= \beta c_1 (N_1 - Y_1) \left[\rho_{11}\left(\frac{Y_1}{N_1}\right) + \rho_{12}\left(\frac{Y_2}{N_2}\right) + \rho_{13}\left(\frac{Y_3}{N_3}\right)\right] - (\nu + \mu) Y_1 \\
&= 0.8c_1 (13644600 - Y_1) \\
&\quad \times \left(2.71666 \times 10^{-8} Y_1 + 1.89183 \times 10^{-8} Y_2 + 4.31015 \times 10^{-9} Y_3\right) - 2.025 Y_1 \\
&= f(Y_1, Y_2, Y_3), \\
Y_2' &= \beta c_2 (N_2 - Y_2) \left[\rho_{21}\left(\frac{Y_1}{N_1}\right) + \rho_{22}\left(\frac{Y_2}{N_2}\right) + \rho_{23}\left(\frac{Y_3}{N_3}\right)\right] - (\nu + \mu) Y_2 \\
&= 0.8c_2 (22031908 - Y_2) \\
&\quad \times \left(1.36916 \times 10^{-8} Y_1 + 2.41527 \times 10^{-8} Y_2 + 5.70016 \times 10^{-9} Y_3\right) - 2.025 Y_2 \\
&= g(Y_1, Y_2, Y_3), \\
Y_3' &= \beta c_3 (N_3 - Y_3) \left[\rho_{31}\left(\frac{Y_1}{N_1}\right) + \rho_{32}\left(\frac{Y_2}{N_2}\right) + \rho_{33}\left(\frac{Y_3}{N_3}\right)\right] - (\nu + \mu) Y_3 \\
&= 0.8c_3 (49305973 - Y_3) \\
&\quad \times \left(5.44930 \times 10^{-9} Y_1 + 1.59584 \times 10^{-8} Y_2 + 1.16533 \times 10^{-8} Y_3\right) - 2.025 Y_3 \\
&= h(Y_1, Y_2, Y_3).
\end{aligned} \tag{3.3}$$

3.2. Analysis and Results

Once again, we consider values of c_1, c_2, and c_3, under the condition that $c_1 \geq c_2 \geq c_3 \geq 1.0$ (Chandra et al.). The analysis with each set of (c_1, c_2, c_3) is the same as in the age stratification.

As in the age stratification, a bifurcation emerged. The points in Table 3.2 represent the largest values of c_i such that the disease-free equilibrium is locally asymptotically stable, where the eigenvalues of the Jacobian matrix are all with negative real parts. From these critical points with three group contact rates, if at least one contact rate is raised, then the endemic equilibrium becomes locally asymptotically stable, which predicts the persistence of the disease. The method in Appendix A was then refined to 0.01 increments in Python, with the resulting c_i values plotted in Figure (3.1), along with the surface of best fit. Visually, any (c_1, c_2, c_3) less than or equal to the surface in Figure (3.1) predicts the asymptotic eradication of gonorrhea.

4. Conclusion

In this paper, we presented a mathematical analysis of an epidemic model which represents the transmission of gonorrhea through core and noncore groups. We focused our analysis on two stratifications: age and education. We determined the approximated average numbers of partners per year for each of our sexual activity groups such that a bifurcation occurred between the disease-free and endemic equilibriums, resulting in values critical for education and prevention measures.

For the age stratification, we determined the number of sexually active people in each of our core and noncore groups in the United States, and then determined the likelihood of individuals from each group mating amongst themselves and amongst those in the other groups. The approximated points at which the bifurcation occurs for the age stratification can be seen in Table 2.2. Similarly, we approached the model using an education stratification. After determining the values

TABLE 3.2. Eradication Points for Education Stratification.

c_1	c_2	c_3	c_1	c_2	c_3	c_1	c_2	c_3	c_1	c_2	c_3	c_1	c_2	c_3
5.9	1.0	1.0	4.6	2.2	1.2	4.0	2.5	1.4	3.5	3.0	1.0	3.1	3.0	1.4
5.8	1.1	1.1	4.6	2.1	1.4	4.0	2.4	1.6	3.5	2.9	1.2	3.1	2.9	1.6
5.7	1.2	1.2	4.6	2.0	1.6	4.0	2.3	1.8	3.5	2.8	1.4	3.1	2.8	1.8
5.6	1.3	1.2	4.6	1.9	1.8	4.0	2.2	1.9	3.5	2.7	1.6	3.1	2.7	1.9
5.5	1.5	1.0	4.5	2.3	1.1	4.0	2.1	2.1	3.5	2.6	1.8	3.1	2.6	2.1
5.5	1.4	1.2	4.5	2.2	1.3	3.9	2.7	1.1	3.5	2.5	1.9	3.1	2.5	2.2
5.5	1.3	1.3	4.5	2.1	1.5	3.9	2.6	1.3	3.5	2.4	2.1	3.1	2.4	2.3
5.4	1.6	1.0	4.5	2.0	1.7	3.9	2.5	1.5	3.5	2.3	2.2	3.1	2.3	2.3
5.4	1.5	1.2	4.5	1.9	1.9	3.9	2.4	1.7	3.4	3.0	1.1	3.0	3.0	1.5
5.4	1.4	1.4	4.4	2.4	1.0	3.9	2.3	1.9	3.4	2.9	1.3	3.0	2.9	1.7
5.3	1.6	1.2	4.4	2.3	1.3	3.9	2.2	2.0	3.4	2.8	1.5	3.0	2.8	1.8
5.3	1.5	1.4	4.4	2.2	1.5	3.9	2.1	2.1	3.4	2.7	1.7	3.0	2.7	2.0
5.2	1.7	1.2	4.4	2.1	1.7	3.8	2.8	1.0	3.4	2.6	1.8	3.0	2.6	2.1
5.2	1.6	1.4	4.4	2.0	1.9	3.8	2.7	1.2	3.4	2.5	2.0	3.0	2.5	2.3
5.2	1.5	1.5	4.3	2.5	1.0	3.8	2.6	1.4	3.4	2.4	2.1	3.0	2.4	2.4
5.1	1.8	1.2	4.3	2.4	1.2	3.8	2.5	1.6	3.4	2.3	2.3	2.9	2.9	1.7
5.1	1.7	1.4	4.3	2.3	1.4	3.8	2.4	1.8	3.3	3.1	1.0	2.9	2.8	1.9
5.1	1.6	1.6	4.3	2.2	1.6	3.8	2.3	2.0	3.3	3.0	1.2	2.9	2.7	2.1
5.0	1.9	1.1	4.3	2.1	1.8	3.8	2.2	2.1	3.3	2.9	1.4	2.9	2.6	2.2
5.0	1.8	1.4	4.3	2.0	2.0	3.7	2.8	1.1	3.3	2.8	1.6	2.9	2.5	2.3
5.0	1.7	1.6	4.2	2.5	1.1	3.7	2.7	1.4	3.3	2.7	1.8	2.9	2.4	2.4
4.9	2.0	1.1	4.2	2.4	1.3	3.7	2.6	1.6	3.3	2.6	1.9	2.8	2.8	2.0
4.9	1.9	1.3	4.2	2.3	1.5	3.7	2.5	1.7	3.3	2.5	2.1	2.8	2.7	2.1
4.9	1.8	1.5	4.2	2.2	1.7	3.7	2.4	1.9	3.3	2.4	2.3	2.8	2.6	2.3
4.9	1.7	1.7	4.2	2.1	1.9	3.7	2.3	2.1	3.2	3.1	1.1	2.8	2.5	2.4
4.8	2.1	1.0	4.2	2.0	2.0	3.7	2.2	2.2	3.2	3.0	1.3	2.7	2.7	2.2
4.8	2.0	1.3	4.1	2.6	1.0	3.6	2.9	1.1	3.2	2.9	1.5	2.7	2.6	2.3
4.8	1.9	1.5	4.1	2.5	1.3	3.6	2.8	1.3	3.2	2.8	1.7	2.7	2.5	2.4
4.8	1.8	1.7	4.1	2.4	1.5	3.6	2.7	1.5	3.2	2.7	1.8	2.6	2.6	2.4
4.7	2.2	1.0	4.1	2.3	1.7	3.6	2.6	1.7	3.2	2.6	2.0	2.6	2.5	2.5
4.7	2.1	1.2	4.1	2.2	1.8	3.6	2.5	1.8	3.2	2.5	2.1			
4.7	2.0	1.4	4.1	2.1	2.0	3.6	2.4	2.0	3.2	2.4	2.3			
4.7	1.9	1.6	4.0	2.7	1.1	3.6	2.3	2.1	3.2	2.3	2.3			
4.7	1.8	1.8	4.0	2.6	1.2	3.6	2.2	2.2	3.1	3.1	1.2			

for our parameters, we analyzed the model and determined the approximated points at which the bifurcation occurred, as seen in Table 3.2.

Through this analysis, we determined the highest average number of partners per year for each age group and education level such that the disease-free equilibrium became asymptotically stable, thereby conjecturing the values ensuring the eradication of gonorrhea. Future work in this area might include an analysis of the global stability for the disease-free equilibrium through method

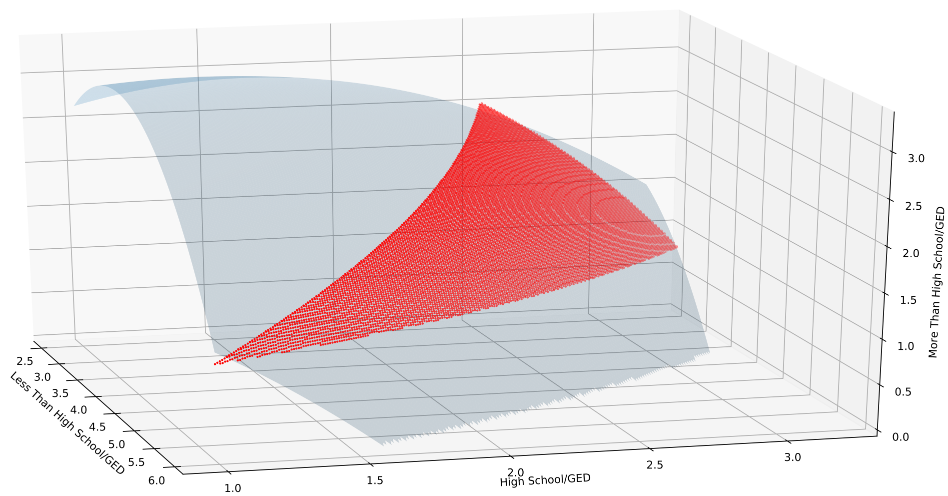

FIGURE 3.1. Approximate Bifurcation Points for Average Partners per Year in Education Stratification with 0.01 Increments.

of upper-lower solutions, as well as the implementation of this data into educational materials by public health departments.

Appendix A. Psuedo-Code for Varying c_i

(1) Let c_1^* be the maximum value of c_1 where $c_1 = c_1^*, c_2 = 1.0, c_3 = 1.0$ results in the stability of the disease-free equilibrium.
(2) Increment c_2 by 0.1 until the disease-free equilibrium is no longer stable. Let c_2^* be the maximum value of c_2 where $c_1 = c_1^*, c_2 = c_2^*, c_3 = 1.0$ results in the stability of the disease-free equilibrium.
(3) Increment c_3 by 0.1 until the disease-free equilibrium is no longer stable. Let c_3^* be the maximum value of c_3 where $c_1 = c_1^*, c_2 = c_2^*, c_3 = c_3^*$ results in the stability of the disease-free equilibrium. This is then a point at which the bifurcation occurs.
(4) Reset c_3^* to 1.0. Then, decrement c_2^* by 0.1.
(5) Repeat steps 3 and 4 until $c_2^* = c_3^*$.
(6) Reset $c_2^*, c_3^* = 1.0$. Then, decrement c_3^* by 0.1.
(7) Repeat steps 2 through 6 until $c_1^* = c_2^* = c_3^*$.

Acknowledgments

The authors would like to thank the anonymous referees for their constructive suggestions on improving the quality of this article.

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