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# Analysis of a Vector-Borne Diseases Model with a Two-Lag Delay Differential Equation

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ABSTRACT. We are concerned with the stability analysis of equilibrium solutions for a two-lag delay differential equation which models the spread of vector-borne diseases, where the lags are incubation periods in humans and vectors. We show that there are some values of transmission and recovery rates for which the disease dies out and others for which the disease spreads into an endemic. The proofs of the main stability results are based on the linearization method and the analysis of roots of transcendental equations. We then simulate numerical solutions using MATLAB. We observe that the solution could possess chaotic and sometimes unbounded behaviors.

## 1. Introduction

Consider the two-lags delay differential equation

$$I'(t) = b\,(1 - eI(t-\tau_h))\,I(t-\tau_h-\tau_v) - cI(t), \qquad t > 0 \tag{1.1}$$

where $b$, $c$, $e$, $\tau_h$ and $\tau_v$ are positive constants. Equation (1.1) models the spread of vector-borne disease where $I$ represents the proportion of infected humans in the population, and $\tau_h$ and $\tau_v$ represent the incubation (delay) period in humans and in vectors, respectively.

Equation (1.1) belongs to the class of delay differential equations (DDEs). These equations are differential equations in which the time derivatives of an unknown function depend on its past values and possibly its derivatives. Models based on delay differential equations have been increasingly used in several fields such as mechanical engineering, chemistry, biology, optics, and epidemiology (see (Erneux, 2009) for more examples and applications). In epidemiology, the spread of infectious diseases is sometimes modeled using DDE's and different types of delay can be considered; examples include but are not limited to incubation and immunity periods.

Vector-borne diseases (VBD's) is a type of infectious diseases that are modeled using DDEs. These diseases are transmitted to humans through the bites of an infected arthropod (e.g. mosquitoes). Malaria and the Zika virus are two well-known examples. Understanding the spread of such diseases is vital to their eventual containment and eradication, and many papers have been devoted to the study of models of vector-borne diseases such as Malaria and the Zika Virus. We refer for instance to Cooke (1979), van den Driessche (1994), Cooke and van den Driessche (1996), Xiao and Zou (2013), Cai et al. (2017), among others. In 1979, Cooke conducted a stability analysis of equilibrium solutions for Equation (1.1), a DDE model that governs the dynamics of a vector-borne disease where a single time lag was considered, namely, the incubation delay period in vectors $\tau_v$. Whereas van den Driessche (1994) surveyed different epidemiological models based on systems

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of DDEs where the delays considered are immunity and incubation periods. In 1996, Cooke and Van Den Driessche established and studied an SEIRS disease transmission model that includes two delays of immunity and incubation. Xiao and Zou (2013) formulated a Malaria infections disease model that is based on a system of DDEs with two distributed delays and provided a detailed analysis in the special cases of having no delays and having two discrete delays. In Cai et al. (2017), the authors consider a VBD model based on a system of DDE's with two incubation periods as distributed delays, and they study the stability of its equilibrium solutions. For (scalar) two-lag delay differential equations, we refer to Braddock and van den Driessche (1983) and the references therein. Braddock and van den Driessche (1983) investigated two-lag DDEs for some biological models, where the two time constant lags are regeneration and reproductive lags.

The purpose of this paper is to extend vector-borne diseases model (Cooke, 1979) by including an additional incubation period (denoted $\tau_h$) that corresponds to humans. We are interested in the stability analysis equilibrium solutions (the disease-free equilibrium and the endemic equilibrium) of Eq. (1.1). Our approach is based on the linearization method and the analysis of roots of transcendental equations of the form

$$z + a_1 + a_2 e^{-z} = 0,$$

where $a_1$ and $a_2$ are real constants. We shall point out that, Hayes (1950) has given conditions on the negativity of $Re(z)$ $\forall z \in \mathbb{C}$ where $z$ is a root of the above transcendental equation. Hayes' approach is based on looking at the polar form of the complex roots and parametrized curves in the complex plane. However, the approach we will present here is more in line with (Erneux, 2009, chap. 2.1) where only the case $a_1 = 0$ and $a_2 \neq 0$ was treated. Herein, we consider the more general case, that is, where $a_1$ and $a_2$ are not necessarily zero. We parametrize $a_2$ and $Re(z)$ in terms of $Im(z)$ and plot the parametrized curves in a $(a_2, Re(z))$-plane with arbitrary $a_1 \in \mathbb{R}$. We also make use of Smith's results on transcendental equations (Smith, 2011, see). In what follows we will prove the following proposition and main theorems.

**Proposition 1.1.** *Consider the following transcendental equation*

$$z + a_1 + a_2 e^{-z} = 0 \tag{1.2}$$

*Then, for all complex roots $z$, $Re(z) < 0$ if and only if the following two conditions hold*

*(1)* $a_1 > -1$

*(2)* $-a_1 < a_2 < V \csc V$ *or equivalently* $-a_1 < a_2 < (V^2 + a_1^2)^{\frac{1}{2}}$ *where* $V \in (0, \pi)$ *is the unique root of* $a_1 = V \cot V$.

*Moreover,* (1.2) *has exactly one real root $z = 0$ with all other roots having negative real parts if and only if $a_1 \geq -1$ and $-a_1 = a_2$. And there are exactly two purely imaginary roots with all other roots having negative real parts if and only if $a_1 > -1$ and $a_2 = V \csc V = (V^2 + a_1^2)^{\frac{1}{2}}$ where $0 < V < \pi$ is the unique root of $a_1 = V \cot V$.*

**Theorem 1.2.** *Consider the initial value problem*

$$\begin{cases} I'(t) = b\,(1 - eI(t - \tau_h))\, I(t - \tau_h - \tau_v) - cI(t), & t > 0 \quad \text{(1.3a)} \\ I(t) = \phi(t), & t \in [-\max\{\tau_v, \tau_h\}, 0] \quad \text{(1.3b)} \end{cases}$$

*If the disease's reproduction number $R_0 = \frac{b}{c} \leq 1$, then the disease-free equilibrium of (1.3a) ($I^* = 0$) is locally stable. Moreover, if $R_0 > 1$, then it is unstable.*

**Theorem 1.3.** *Assume that $\tau_v = 0$. If $0 \leq R_0 < 1$, then the endemic equilibrium of (1.3a) ($I^* = \frac{1}{e}\left(1 - \frac{c}{b}\right)$) is unstable. If $R_0 = 1$, then it coincides with the locally stable disease-free equilibrium. If $1 < R_0 < \frac{b_0}{c}$, it is locally stable. If $R_0 \geq \frac{b_0}{c}$, then it is unstable, where $b_0 = \frac{1}{\tau_h}\left[(V^2 + \tau_h^2 c^2)^{\frac{1}{2}} + 2\tau_h c\right]$ is a critical transmission number and $V \in (0, \pi)$ is the unique solution of $\tau_h c = V \cot V$.*

We organize the paper as follows. In Section 2, we give the general assumptions and establish the model of a scalar two-lag delay differential equation that governs the dynamics of the infected human population. In Section 3, we discuss the stability of equilibrium solutions of problem (1.3a)–(1.3b) and prove our main results. Finally, Section 4 is devoted to MATLAB numerically simulated solutions for which the code is given at the end of the paper; we provide observations on boundedness and general behavior of the solution. Visual renditions are included throughout the paper.

## 2. Two-Lag Vector-Borne Diseases Model

In this section, we introduce some notation and assumptions and establish the model.

### 2.1. Notation

In what follows, we make use of the following notation:

$t =$ Time unit
$S_h(t) =$ Number of susceptible humans in the population
$I_h(t) =$ Number of infected humans in the population
$E_h(t) =$ Number of exposed humans in the population
They are the humans that are currently developing the disease
$N_h(t) =$ Total number of humans
$I_v(t) =$ Number of infected vectors
$\beta =$ Biting rate of vectors, that is the number of humans bitten by one vector per time unit
$c =$ Human recovery rate, that is the fraction of the infected group that recovers per unit time
$\tau_v =$ Incubation period in vectors, that is the duration of exposure of vectors
$\tau_h =$ Incubation period in humans, that is the duration of exposure of humans

### 2.2. Assumptions

The model is based on the following set of assumptions:

(1) Upon biting an infected human with a biting rate $\beta \geq 0$, a susceptible vector becomes infected. And when a susceptible human gets bitten by an infected vector, at the same rate, the bitten human becomes exposed.
(2) Infected humans recover from the disease at rate $c \geq 0$ and they confer no immunity after recovery.

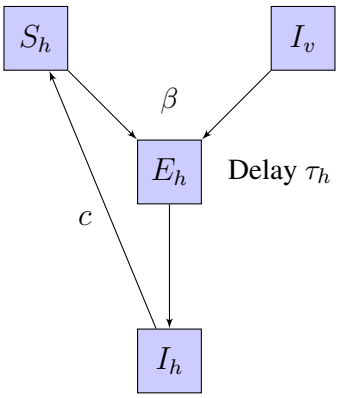


FIGURE 2.1. Compartmental Model

(3) There is an incubation period $\tau_h$ in humans and an incubation period $\tau_v$ in vectors, and they constitute the delays of the model.
(4) The size of the human population $N_h$ is fixed and each human can either be susceptible, exposed, or infected (i.e. $S_h + I_h + E_h = N_h$).
(5) The infected vector population is proportional to the infected human population, that is $I_v(t) = pI_h(t - \tau_v)$.
(6) The exposed human population is proportional to the infected human population, that is $E_h(t) = qI_h(t)$.

Assumption (5) follows by assuming a very large vector population size, so that the infected vector population would simply be proportional to the infected human population. As for assumption (6), it follows by assuming that the size of the exposed human population is simply a linear function of the size of the infected human population.

### 2.3. The Equation

We are interested in a scalar delay differential equation that governs the dynamics of the infected human population $I_h$. From assumptions $(1)$, $(2)$, and $(3)$, we derive the equation

$$I_h'(t) = \beta \frac{S_h(t-\tau_h)}{N_h(t-\tau_h)} I_v(t-\tau_h) - cI_h(t) \tag{2.1}$$

To have $I_h$ as the only dependent variable, we apply assumptions $(4)$, $(5)$, and $(6)$ to rewrite (2.1) as follows

$$\begin{aligned} I_h'(t) &= \beta \frac{N_h - I_h(t-\tau_h) - E_h(t-\tau_h)}{N_h} I_v(t-\tau_h) - cI_h(t) \\ &= \beta p \left(1 - \frac{I_h(t-\tau_h)}{N_h} - q\frac{I_h(t-\tau_h)}{N_h}\right) I_h(t-\tau_h-\tau_v) - cI_h(t) \end{aligned}$$

Dividing by $N_h$ and rewriting, we get the model's equation:

$$\begin{cases} I'(t) = b\left(1 - eI(t-\tau_h)\right) I(t-\tau_h-\tau_v) - cI(t), & t > 0, \\ I(t) = \phi(t), & t \in [-\max\{\tau_v, \tau_h\}, 0]. \end{cases}$$

where $b = \beta p$ is what we refer to as the transmission number, $e = 1+q$, $I = \frac{I_h}{N_h}$ is the proportion of the human infected group in the population, and $\phi(t) : [-\max\{\tau_v, \tau_h\}, 0] \to [0, 1]$ is an integrable function of time corresponding to the initial infected human proportion.

Observe that if $\tau_h = 0$, $e = 1$, and $\tau_v \neq 0$, then we have the model studied by Cooke (1979).

## 3. Stability Analysis and Proofs of the Main Results

In this section, we study the local stability of the equilibrium solutions of Eq.(1.3a)-(1.3b) to determine disease-free and endemic dynamics.

The equilibrium solutions of (1.3a) are its constant time-independent solutions $I^*$ that satisfy the equation

$$b\left(1 - eI^*\right) I^* - cI^* = 0 \tag{3.1}$$

Solving the algebraic equation (3.1), we get the equilibrium solutions

(1) $I^* = 0$ (the disease-free equilibrium)

(2) $I^* = \dfrac{1}{e}\left(1 - \dfrac{c}{b}\right)$ (the endemic equilibrium which exists when $\frac{b}{c} \geq 1$)

Applying a small perturbation $u(t)$ around the disease-free equilibrium and linearizing yield

$$u'(t) = bu(t - \tau_h - \tau_v) - cu(t) \tag{3.2}$$

And linearizing around the endemic equilibrium gives the following equation

$$u'(t) = cu(t - \tau_h - \tau_v) + (c - b)u(t - \tau_h) - cu(t) \tag{3.3}$$

In each of the linearized delay differential equations (3.2) and (3.3), we substitute the exponential solution $u(t) = e^{\lambda t}$ into the equations and derive the characteristic equations associated to each equilibrium. The characteristic equation corresponding to (3.2) is given by

$$\lambda = be^{(-\tau_v - \tau_h)\lambda} - c \tag{3.4}$$

And the characteristic equation corresponding to the endemic equilibrium is given by

$$\lambda + c = ce^{(-\tau_h - \tau_v)\lambda} + (c - b)e^{-\tau_h \lambda} \tag{3.5}$$

In order to prove Theorems 1.2 and 1.3, we first need to analyze the characteristic equations (3.4) and (3.5). Observe that equation (3.4) has the form of equation (1.2) and assuming $\tau_v = 0$, (3.5) can be transformed to equation (1.2). We note that since our focus lies on analyzing the transcendental equation (1.2), we assume $\tau_v$ is zero when looking at the stability of the endemic equilibrium.

We will first present an approach to characterizing roots of transcendental equations in $z$ that take the form:

$$z + a_1 + a_2 e^{-z} = 0 \tag{1.2}$$

where $a_1$ and $a_2$ are real parameters not necessarily zero and $z$ is a complex root. We specifically characterize the real parts of $z$ with respect to parameters $a_1$ and $a_2$, and we seek having $Re(z) < 0$ for all complex roots $z$ to achieve stability.

Equation (1.2) has been studied by several authors; we refer to Hayes (1950), Smith (2011), and Erneux (2009). Hayes (1950) has provided a complete characterization of the root curves of (1.2); his approach was mainly based on looking at the polar form of the complex roots and illustrating his characterization with the complex plane. Smith (2011, chap. 4.5) also gave a similar characterization. In this paper, we use a different approach which is in line with Erneux's (2009, chap. 2.1) who dealt with the case of $a_1 = 0$. It is based on parametrizing $a_2$ and $Re(z)$ in

terms of $Im(z)$ and plotting the parametrized curves in a $(a_2, Re(z))$-plane. We assume arbitrary $a_1 \in \mathbb{R}$.

First, we state some known results regarding the following transcendental equation

$$\lambda + \beta e^{-\lambda} = 0 \tag{3.6}$$

where $\beta$ is a real parameter and $\lambda = x \pm iy$ with $y \geq 0$ (note that if we have a complex root then its conjugate is a root as well). Then, we illustrate the parametrization of Erneux (2009, chap. 2.1) and prove a more general result on its curves. Then, we prove the main result regarding negativity of $Re(z)$ in (1.2).

Note that equation (3.6) can be rewritten as a system when setting $\lambda = x \pm iy$ and using Euler's formula. We obtain the following equation

$$x + iy = (-\beta e^{-x} \cos y) + i(\beta e^{-x} \sin y)$$

Therefore, we have

$$\begin{cases} x = -\beta e^{-x} \cos y & \text{(3.7a)} \\ y = \beta e^{-x} \sin y & \text{(3.7b)} \end{cases}$$

From (3.7b), we observe that $y = 0$ if and only if $\sin y = 0$.

Let us state some auxiliary lemmas that will be needed in the proof of proposition 3.6. Their proofs can be found in Smith (2011):

**Lemma 3.1.** *(Smith, 2011, Lemma 2.2)*
*For $\lambda \in \mathbb{R}$:*

*(1) If $\beta < 0$, then (3.6) has exactly one positive real root $\lambda_0$.*
*Moreover, $\lambda_0 \to 0$ as $\beta \to 0^-$ and $\lambda_0 \to \infty$ as $\beta \to -\infty$.*
*(2) If $0 < \beta < e^{-1}$, then it has exactly two negative real roots $\lambda_1$ and $\lambda_2$ where $\lambda_1 < -1$ and $-1 < \lambda_2 < 0$. Moreover, $\lambda_2 \to 0$ and $\lambda_1 \to -\infty$, as $\beta \to 0$, and $\lambda_2 \to -1$ and $\lambda_1 \to -1$, as $\beta \to e^{-1}$.*
*(3) If $\beta = e^{-1}$, it has one negative real root: $\lambda = -1$.*
*(4) If $\beta > e^{-1}$, then there are no real roots.*

**Lemma 3.2.** *(Smith, 2011, Proposition 2.1) If $0 < \beta < \frac{\pi}{2}$, then all roots of (3.6) have negative real parts.*

**Lemma 3.3.** *(Smith, 2011, Lemma 2.3)*
*If $\frac{1}{e} < \beta < \frac{\pi}{2}$, then there exists a pair of complex conjugate roots $\lambda = x \pm iy$ to (3.6) where $-1 < x < 0$ and $0 < y < \frac{\pi}{2}$. The $\beta$'s and $x$'s that correspond to the roots can be parametrized by $y$:*

$$\beta = \beta(y),\ x = x(y)$$

*where $\beta(y)$ and $x(y)$ are increasing functions of $y$ where $\beta(y)$ is positive and $x(y)$ is negative. Moreover, $x \to -1$, $\beta \to \frac{1}{e}$, as $y \to 0$, and $x \to 0$, $\beta \to \frac{\pi}{2}$, as $y \to \frac{\pi}{2}$.*

**Lemma 3.4.** *(Smith, 2011, Proposition 2.1)*
*If $\beta > \frac{\pi}{2}$, then there exists a pair of complex conjugate roots $\lambda = x \pm iy$ to (3.6) where $x > 0$ and $\frac{\pi}{2} < y < \pi$. For this pair, $\beta$, $y$, and $x$ can be parametrized by $\theta = \arg(\lambda)$:*

$$\beta = \beta(\theta), x = x(\theta), y = y(\theta)\ \ 0 < \theta < \frac{\pi}{2}$$

*where $\beta(\theta)$, $x(\theta)$, and $y(\theta)$ are all decreasing functions of $\theta$. Moreover, $y \to \pi, x \to +\infty, \beta \to +\infty$ as $\theta \to 0^+$ and $y \to \frac{\pi}{2}^+, x \to 0^+, \beta \to \frac{\pi}{2}^+$ as $\theta \to \frac{\pi}{2}^-$.*

To visually illustrate why the above statements are true, we use the parametrization in (Erneux, 2009, chap 2.1). Assuming non-real roots, that is assuming $y \neq 0$ which by (3.7b) equivalently implies $\sin y \neq 0$, and dividing (3.7a) by (3.7b), we get $x = -y\cot(y)$. Substituting that in the second equation, we get $\beta = \frac{y}{e^{y\cot(y)}\sin(y)}$. We thus obtain the following $(\beta(y), x(y))$ parametrization

$$\begin{cases} x = -y\cot(y) & \text{(3.8a)} \\ \beta = \dfrac{y}{e^{y\cot(y)}\sin(y)} & \text{(3.8b)} \end{cases}$$

In Figure 3.1 below, the solid styled curves/strips correspond to the graph described by equations (3.8a)-(3.8b). It relates $\beta$ to the real part $x$ of the complex non-real roots of (3.6). The dashed curve relates $\beta$ to real roots of (3.6). It corresponds to the equation $\beta = -xe^x$ ($y = 0$). Points on the graph of Figure 3.1 that lie on the same vertical line correspond to one value of $\beta$ and each would correspond to a conjugate pair of roots with real part $x$, represented by the ordinate/height of the point. Moreover, the plot in Figure 3.1 provides an illustration for Lemmas 3.1, 3.2, 3.3, and 3.4. Notice that the intersection between the $\beta$-axis and the highest solid curve in the right-half plane is achieved at $\beta = \frac{\pi}{2}$.

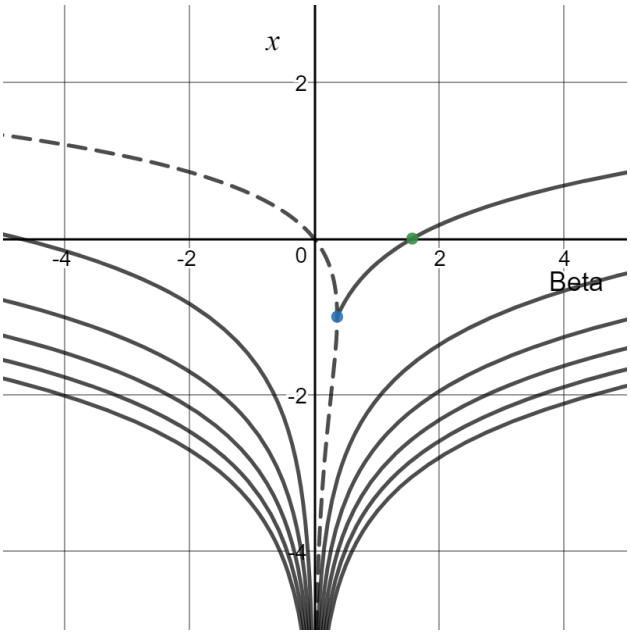


FIGURE 3.1. $(\beta, x)$-plane plot corresponding to the complex (solid-style) and real (dashed-style) roots of (3.6)

Before proving the main result on the strips of Figure 3.1, we prove the following lemma which shows that two strips (real or non-real) do not intersect:

**Lemma 3.5.** *Let $\lambda_1 = x_1 \pm iy_1$ and $\lambda_2 = x_2 \pm iy_2$ with $y_1, y_2 > 0$ be two non-real roots of (3.6) that correspond to a fixed $\beta$. Then, $x_1 = x_2$ if and only if $y_1 = y_2$. Moreover, if $\lambda_1$ is real (i.e. $y_1 = 0$) while $y_2 > 0$, then $x_1 \neq x_2$.*

***Proof.*** Consider non-real roots $\lambda_1$ and $\lambda_2$. Suppose that $y_1 = y_2$. Then, by (3.8a), we have

$$x_1 = -y_1 \cot y_1 = -y_2 \cot y_2 = x_2.$$

On the other hand, suppose that $x_1 = x_2$. Since the two root pairs correspond to a fixed $\beta$, by (3.7a), we have $\cos y_1 = \cos y_2$. This gives $\sin y_1 = \pm \sin y_2$. Now, since $y_1, y_2 > 0$, then by (3.7b), $\sin y_1$ and $\sin y_2$ share the same sign so that $\sin y_1 = \sin y_2$. From $(3.7b)$, this gives $y_1 = y_2$.

Now, consider real $\lambda_1$ and non-real $\lambda_2$. We seek to show that $x_1 \neq x_2$. Otherwise, by (3.6) and (3.7a), $-\beta e^{-x_1} = -\beta e^{-x_2} \cos y_2$ which would imply that $\cos y_2 = 1$ giving $\sin y_2 = 0$. Then, by (3.7b), $y_2 = 0$, a contradiction. Hence, $x_1 \neq x_2$.

□

The following proposition provides a complete characterization of the complex non-real roots of (3.6) through the lense of the parametrization in terms of the imaginary part $y$.

**Proposition 3.6.** *Given equation (3.6), we have the following:*

*(1) If $\beta > e^{-1}$ and $y \in (0, \pi)$, then there exists a unique pair of complex conjugate roots $\lambda_1 = x \pm iy$ to (3.6) where $x > -1$. The $\beta$'s and $x$'s that correspond to the roots can be parametrized by $y$, and they would be given by*

$$\beta = \beta(y),\ x = x(y),$$

*where $\beta(y)$ and $x(y)$ are increasing functions of $y$, and $\beta(y)$ is positive.*
*Moreover, $x \to -1$, $\beta \to \frac{1}{e}$ as $y \to 0$; $x \to \infty$, $\beta \to \infty$ as $y \to \pi$; and $\beta(\frac{\pi}{2}) = \frac{\pi}{2}$.*
*Further, when $0 < y < \frac{\pi}{2}$, $x(\frac{\pi}{2}) < 0$; when $y = \frac{\pi}{2}$, $x(\frac{\pi}{2}) = 0$; and when $\frac{\pi}{2} < y < \pi$, $x(y) > 0$.*

*(2) If $\beta > 0$ and $y \in ((2n-2)\pi, (2n-1)\pi)$ for $n \in \mathbb{N}_{\geq 2} = \{n \in \mathbb{N} : n \geq 2\}$, then there exists a unique pair of complex conjugate roots $\lambda_n = x \pm iy$ to (3.6) where $x \in \mathbb{R}$. The $\beta$'s and $x$'s that correspond to the roots can be parametrized by $y$, and they would be given by*

$$\beta = \beta(y),\ x = x(y),$$

*where $\beta(y)$ and $x(y)$ are strictly increasing functions of $y$.*
*Moreover, $x \to -\infty$, $\beta \to 0$ as $y \to (2n-2)\pi$; $x \to \infty$, $\beta \to \infty$ as $y \to (2n-1)\pi$; and $\beta(\frac{4n-3}{2}\pi) = \frac{4n-3}{2}\pi$.*
*Further, when $(2n-2)\pi < y < \frac{4n-3}{2}\pi$, $x(y) < 0$; when $y = \frac{4n-3}{2}\pi$, $x(y) = 0$; and when $\frac{4n-3}{2}\pi < y < (2n-1)\pi$, $x(y) > 0$.*

*(3) For each $\beta < 0$ and when $y$ is restricted to $((2n-1)\pi, 2n\pi)$ for $n \in \mathbb{N}_{\geq 1}$, there exists a unique pair of complex conjugate roots $\lambda_n = x \pm iy$ to (3.6) where $x \in \mathbb{R}$. The $\beta$'s and $x$'s that correspond to the roots can be parametrized by $y$, and they would be given by*

$$\beta = \beta(y),\ x = x(y),$$

*where $\beta(y)$ and $x(y)$ are respectively strictly decreasing and strictly increasing functions of $y$.*
*Moreover, $x \to -\infty$, $\beta \to 0$ as $y \to (2n-1)\pi$; $x \to \infty$, $\beta \to -\infty$ as $y \to 2n\pi$;*
*and $\beta(\frac{4n-1}{2}\pi) = \frac{4n-1}{2}\pi$.*
*Further, when $(2n-1)\pi < y < \frac{4n-1}{2}\pi$, $x(y) < 0$; when $y = \frac{4n-1}{2}\pi$, $x(y) = 0$; and when $\frac{4n-1}{2}\pi < y < 2n\pi$, $x(y) > 0$.*

*(4) For each $\beta > 0$, consider the roots $\lambda_n$ determined in the first and second assertion. We have that $Re(\lambda_{n+1}) < Re(\lambda_n)$ for $n \in \mathbb{N}_{\geq 1}$ when $\beta > e^{-1}$ and $n \in \mathbb{N}_{\geq 2}$ when $0 < \beta \leq e^{-1}$. On the other hand, for each $\beta < 0$, consider the roots $\lambda_n$ determined in the third assertion. We have that $Re(\lambda_{n+1}) < Re(\lambda_n)$ for $n \in \mathbb{N}_{\geq 1}$.*

*(5) For $\beta < 0$, the highest strip (a.k.a. the principal strip) is the one corresponding to the positive real root of (3.6). For $0 < \beta \leq e^{-1}$, it is the one corresponding to the larger of the two negative real roots. And for $\beta > e^{-1}$, it is the complex non-real strip corresponding to $y \in (0, \pi)$.*

The first assertion of Proposition 3.6 shows the existence of the complex non-real strip that bifurcates from the real dashed strip at $\beta = e^{-1}$. The second shows the existence of the other complex non-real strips for $\beta > 0$. The third assertion shows the existence of the ones for $\beta < 0$. The fourth assertion gives their ordering with respect to height in the $(\beta, x)-$plane. And the fifth one determines the principal/highest strips in the plane.

***Proof of Proposition 3.6***.

**(1)** This assertion follows from Lemma 3.3 and Lemma 3.4.

**(2)** Restrict $y$ to $S \equiv ((2n-2)\pi, (2n-1)\pi)$ where $n \in \mathbb{N}_{\geq 2}$ for the sake of finding the desired root. First, one can see that $x(y)$ is strictly increasing since

$$x'(y) = (-y\cot(y))' = \frac{y - \frac{1}{2}\sin(2y)}{\sin^2 y} > 0, \quad y \in S.$$

Taking the desired limits on $x(y)$, we get

$$\lim_{y\to(2n-2)\pi} x(y) = \lim_{y\to(2n-2)\pi} (-y\cot y) = -\infty$$

$$\lim_{y\to(2n-1)\pi} x(y) = \lim_{y\to(2n-1)\pi} (-y\cot y) = +\infty$$

Furthermore, $\beta(y)$ is also strictly increasing since

$$\beta'(y) = \left(\frac{y}{\sin(y)}\right)' e^{x(y)} + \frac{y}{\sin(y)} x'(y) e^{x(y)} > 0, \quad y \in S$$

Taking the limits, we get

$$\begin{aligned}\lim_{y\to(2n-2)\pi} \beta(y) &= \lim_{y\to(2n-2)\pi} \frac{y}{\sin y} e^{x(y)} \\ &= \lim_{y\to(2n-2)\pi} \frac{y\cos y}{\sin y\cos y} e^{x(y)} \\ &= \lim_{y\to(2n-2)\pi} -x(y)e^{x(y)}\sec y\end{aligned}$$

Meanwhile,

$$\lim_{y\to(2n-2)\pi} \sec y = 1$$

and noting that $\lim_{y\to(2n-2)\pi} x(y) = -\infty$, we have that

$$\lim_{y\to(2n-2)\pi} -x(y)e^{x(y)} = \lim_{y\to(2n-2)\pi} \frac{-x(y)}{e^{-x(y)}} \overset{\text{LHR}}{=} \lim_{y\to(2n-2)\pi} \frac{-x'(y)}{-x'(y)e^{-x(y)}} = 0.$$

Hence,

$$\lim_{y\to(2n-2)\pi}\beta(y)=0$$

On the other hand,

$$\lim_{y\to(2n-1)\pi}\beta(y)=\lim_{y\to(2n-1)\pi}\frac{y}{\sin(y)}e^{x(y)}=+\infty.$$

Thus, $\beta$ strictly increases from 0 to $+\infty$ and $x$ strictly increases from $-\infty$ to $+\infty$ as $y$ increases from $(2n-2)\pi$ to $(2n-1)\pi$.

Each $\beta\in(0,+\infty)$ corresponds to a unique $y\in S$, and each $y\in S$ corresponds to a unique $x\in\mathbb{R}$. This implies that each $\beta$ corresponds to a unique $\lambda_n=x\pm iy$ where $x\in(-\infty,+\infty)$ and $y\in((2n-2)\pi,(2n-1)\pi)$. As for $\beta(\frac{4n-3}{2}\pi)$, we have that

$$\beta\left(\frac{4n-3}{2}\pi\right)=\frac{\frac{4n-3}{2}\pi e^{x(\frac{4n-3}{2}\pi)}}{\sin(\frac{4n-3}{2}\pi)}=\frac{4n-3}{2}\pi.$$

Lastly, when $(2n-2)\pi<y<\frac{4n-3}{2}\pi$, $x(y)<0$; when $y=\frac{4n-3}{2}\pi$, $x(y)=0$;
and when $\frac{4n-3}{2}\pi<y<(2n-1)\pi$, $x(y)>0$. This is because $\cot y$ in (3.8a) is negative, zero, and positive respectively in each of the three cases.

**(3)** Following similar argument as in the proof of assertion (2), we get the result.

**(4)** For each $\beta>e^{-1}$ and for $n\in\mathbb{N}_{\geq 1}$, we have roots $\lambda_n$ determined in the first and second assertion. We seek to show that $Re(\lambda_{n+1})<Re(\lambda_n)$. We first note that at some fixed $\beta>\frac{1}{e}$, $Re(\lambda_{n+1})\neq Re(\lambda_n)$ by Lemma 3.5 since $Im(\lambda_{n+1})\neq Im(\lambda_n)$.

Hence, using the intermediate value theorem, one can show that either $Re(\lambda_{n+1})<Re(\lambda_n)$ for all $\beta>e^{-1}$ or $Re(\lambda_{n+1})>Re(\lambda_n)$ for all $\beta>e^{-1}$. Thus, it is enough to show that $Re(\lambda_{n+1})<Re(\lambda_n)$ for one $\beta$ to get the desired result. By assertion (1), we know that $Re(\lambda_n)=0$ at $\beta=\frac{4n-3}{2}\pi>e^{-1}$ and $Re(\lambda_{n+1})=0$ at $\beta=\frac{4n-1}{2}\pi$. Due to the invertibility of the strictly increasing functions $\beta(y)$ and $x(y)$ given in the second assertion, we can express $Re(\lambda_{n+1})$ as a strictly increasing function of $\beta$. From this, we derive that $Re(\lambda_{n+1})|_{\beta=\frac{4n-3}{2}\pi}<Re(\lambda_{n+1})|_{\beta=\frac{4n-1}{2}\pi}=0$. Therefore, $Re(\lambda_{n+1})<Re(\lambda_n)$ at $\beta=\frac{4n-3}{2}\pi$; hence, the inequality extends to each $\beta>e^{-1}$. The proof of the rest of the assertion follows a similar argument.

**(5)** Consider $\beta<0$. By Lemma 3.1, there exists only one positive real root $\lambda_1=x_1$. To show that all non-real roots have smaller real parts, suppose for the sake of contradiction that there exists a complex non-real root $\lambda_2=x_2\pm y_2 i$ with $x_2\geq x_1$ and $y_2>0$. Observe that $-\beta>0$ and $x_2-x_1\geq 0$; it follows from (3.7a) that

$$x_2=-\beta e^{-x_2}\cos y_2\geq x_1=-\beta e^{-x_1}\implies\cos y_2\geq e^{x_2-x_1}\geq 1$$

$\cos y_2\geq 1$ gives $\cos y_2=1$ which yields $\sin y_2=0$ implying $y_2=0$ by (3.7b), a contradiction.

Now, consider $0<\beta\leq e^{-1}$. It is enough to show that all non-real roots have real parts less than $-1$ since the larger of the two real negative roots, namely $\lambda_1$, is no less than that value (this follows from Lemma 3.1). Suppose for the sake of contradiction that for some $\beta\in(0,e^{-1}]$, there exists a pair of non-real conjugate roots $\lambda_2=x_2\pm y_2 i$ to (3.6) with $x_2\in[-1,0)$ and $y_2>0$. By (3.7b),

$\frac{\sin y_2}{y_2} = \frac{e^{x_2}}{\beta}$. Moreover, since $-1 \leq x_2 < 0$, then $0 < \beta \leq e^{-1} \leq e^{x_2} < 1$. Hence, $\frac{\sin y_2}{y_2} \geq 1$, which doesn't hold when $y_2 > 0$.

Finally, the result for $\beta > e^{-1}$ follows by induction from Assertion $4$.

□

Now that a complete characterization and description of the strips in Figure 3.1 has been given, we are ready to prove Proposition 1.1.

***Proof of Proposition 1.1***.

Set $\lambda = x \pm iy = z + a_1$ and $\beta = a_2 e^{a_1}$. Then, the equation (1.2) becomes

$$\lambda + \beta e^{-\lambda} = 0$$

and $Re(z) < 0$ for all roots $z$ if and only if $Re(\lambda) < a_1$ for all roots $\lambda$. This is equivalent to finding $\beta$'s for which all of the root strips in the $(\beta, x)$-plane lie below the horizontal line $x = a_1$. This is accomplished by studying the relative position of the principal strips determined in the fifth assertion of Proposition 3.6 to the line $x = a_1$ (See Figure 3.2).

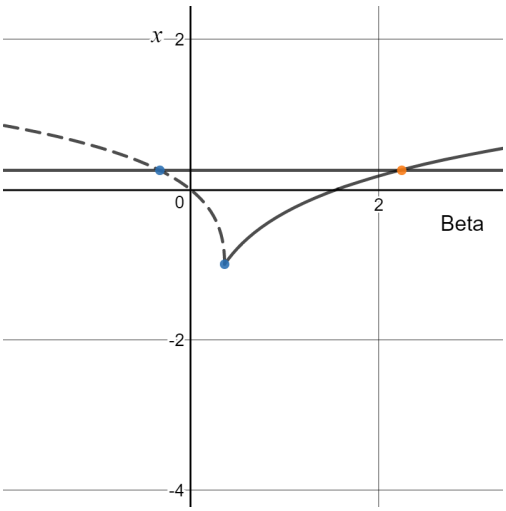


FIGURE 3.2. Graph of (1.2)'s principal roots in the $(\beta, x)$-plane

First, notice that when $a_1 < -1$, the leading strips are higher than the horizontal line $x = a_1$ for all $\beta \in \mathbb{R}$; when $a_1 = -1$, the dashed strip touches $x = a_1$ at $\beta = e^{-1}$; and the two strips lie above the line $x = a_1$ for all other values of $\beta$. Hence, $a_1 > -1$ is a necessary condition to have $Re(\lambda) < a_1$ for all roots $\lambda$.

Now, if $a_1 > -1$, then the line $x = a_1$ intersects each principal strip in a unique point due to the monotonicity of their parametrizations given in Proposition 3.6. It follows from equation (3.6) that the dashed strip corresponding to real roots intersects the line $x = a_1$ at $\beta = -a_1 e^{a_1}$. Now, consider the solid principal strip which was determined in Assertion 5 of Proposition 3.6 and which corresponds to $y \in (0, \pi)$. For $x = a_1$, $a_1 = -y \cot y$ has a unique root $V \in (0, \pi)$. Plugging $y = V$ into (3.8a), we get $\beta = e^{a_1} V \csc V$. Hence, the second necessary condition is to have $-a_1 e^{a_1} < \beta < e^{a_1} V \csc V$ so that both strips would lie below the line $x = a_1$. By plugging

$\beta = e^{a_1}a_2$ back into the second condition, it becomes $-a_1 < a_2 < V\csc V$.

Moreover, $-a_1 < a_2 < V\csc V$ is equivalent to $-a_1 < a_2 < (V^2 + a_1^2)^{\frac{1}{2}}$ since

$$\begin{aligned}(V^2 + a_1^2)^{\frac{1}{2}} &= (V^2 + (-V\cot V)^2)^{\frac{1}{2}} \\ &= |V|(1 + \cot^2 V)^{\frac{1}{2}} \\ &= |V\csc V| \\ &= V\csc V, \qquad \text{since } V\csc V > 0 \text{ when } 0 < V < \pi.\end{aligned}$$

As for the rest of the theorem, we only need to deal with the intersections of the line $x = a_1 \geq -1$ with the principal strips. In this case, $a_1 \geq -1$; otherwise, if $a_1 < -1$, then there is at least one root with real part greater than $a_1$. The line $x = a_1$ intersects the dashed strip (where the point of intersection corresponds to $\lambda = a_1$ or $z = 0$) at $\beta = -a_1e^{a_1}$ which is equivalent to $-a_1 = a_2$. And it intersects the solid strip (where we have a pair of complex conjugate roots) at $\beta = e_1^a V\csc V$ which is equivalent to $a_2 = V\csc V$.

□

We now take up to the proofs of Theorems 1.2 and 1.3.

***Proof of Theorem 1.2***.

Consider the disease-free equilibrium of (1.3a). Its characteristic equation (3.4) can be rewritten by setting $z = (\tau_v + \tau_h)\lambda$ so that $Re(\lambda) < 0$ for all roots $\lambda$ of (3.4) if and only if $Re(z) < 0$ for all roots $z$ of the equation

$$z + (\tau_v + \tau_h)c - b(\tau_v + \tau_h)e^{-z} = 0 \tag{3.9}$$

Applying Proposition 1.1 on equation (3.9), we seek to choose parameters for which stability is achieved, and this occurs if the following two conditions hold

(1) $c(\tau_v + \tau_h) > -1$
(2) $-c(\tau_v+\tau_h) < -b(\tau_v+\tau_h) < V\csc V$ where $0 < V < \pi$ is the unique root of $a_1 = V\cot V$.

We see that the first condition is satisfied due to the positivity of $c(\tau_v + \tau_h)$. As for the second condition, the second part of the inequality ($-b(\tau_v + \tau_h) < V\csc V$) is satisfied since $V\csc V > 0$ for $V \in (0, \pi)$ while $-b(\tau_v + \tau_h)$ is non-positive, and the first part ($-c(\tau_v + \tau_h) < -b(\tau_v + \tau_h)$) boils down to $b < c$. Hence, the disease-free equilibrium is locally asymptotically stable if $b < c$ and unstable if $b > c$. Now, if $b = c$, then $c(\tau_v + \tau_h) = -b(\tau_v + \tau_h)$ so that by Proposition 1.1, $z = 0$ is a root with all other roots having negative real parts. The root $z = 0$ corresponds to a constant particular solution to DDE (3.2), specifically the zero solution which would be the sole equilibrium of (3.2).

□

***Proof of Theorem 1.3***.

Consider the endemic equilibrium. Assuming $\tau_v = 0$ and $z = \tau_h\lambda$, equation (3.5) becomes

$$z + \tau_h c - (2c - b)\tau_h e^{-z} = 0. \tag{3.10}$$

By Proposition 1.1, the endemic equilibrium is stable if $\tau_h c > -1$, which is satisfied by non-negativity of $\tau_h$, and if $-\tau_h c < -\tau_h(2c - b) < V\csc V$ where $0 < V < \pi$ is the unique root of

$\tau_h c = V \cot V$.

The inequality $-\tau_h c < -\tau_h(2c - b)$ is satisfied only when $b > c$. This shows that the endemic equilibrium is unstable when $R_0 = \frac{b}{c} < 1$. If $R_0 = 1$, the equilibrium coincides with the disease-free equilibrium which we proved in Theorem 1.2 to be locally stable for $R_0 = 1$. Now, if $R_0 > 1$, then the condition that remains to be met is $-\tau_h(2c - b) < V \csc V$ which translates to $R_0 < \frac{b_0}{c}$ where $b_0 = \frac{1}{\tau_h}\left[(V^2 + \tau_h^2 c^2)^{\frac{1}{2}} + 2\tau_h c\right]$. □

## 4. Numerical Simulations and Observations

The simulated numerical solutions of (1.3a)-(1.3b) (see Figure 4.1) were created by running a MATLAB script (code is given at the end of the paper) that solves the equation and plots the solution for different sets of parameters (see Table 4.1). Note that in all parameter sets, we take $e = 1$, for one can verify that $e$ is merely a vertical scaling to the solution of (1.3a)-(1.3b) when $e = 1$.

For $b < c$, the solution always approaches the stable disease-free equilibrium asymptotically (see Figure 4.1a). As we increase both delays, spikes start to show up (see Figure 4.1b), but we observe no effect on stability and boundedness when delays are varied. For $b = c$, the solution approaches the zero equilibrium but in a slower manner (see Figures 4.1c and 4.1d). Observe that making $\phi$ sinusoidal in Figure 4.1d leads to a sinusoidal pattern showing up in the plot of the solution.

When considering $b > c$, we first fix $\tau_h = 8.5$, $\tau_v = 0$, and $c = 0.11$ to verify the results of Theorem 1.3 (the values chosen for the fixed parameters $\tau_h$ and $c$ are reasonable as they are close to the ones that correspond to the Zika Virus and which can be found in (Agusto et al., 2017)). Note that for these values, $b_0$ evaluates to $0.3415$.

We observe that the endemic equilibrium is stable for values of $b$ smaller than $b_0$ (see Figure 4.1e). As we increase $b$, oscillations begin to grow around the stable endemic equilibrium, then a bifurcation occurs when $b$ crosses the critical value $b_0 = 0.3415$ and we observe a stable periodic solution around the now unstable endemic equilibrium (see Figure 4.1f).

Further increase in $b$ can lead to oscillations that cross the zero-line (see Figure 4.1g), and even further increase can lead to the solution falling in the negative direction after a certain time (see Figure 4.1h). We also make a note on the erratic and uncontrollable behavior of the solution. This is depicted by Figures 4.1i and 4.1j where we set $\tau_v = 2$ ; notice that a very small change in the history function $\phi$ has caused a notable change in the plot around $t = 300$. From the above, we observe that the consistency (i.e. un-erraticity and predictability) of the model is achieved on a range of transmission values $b$ where the width of the range depends on the value of $b_0$. Therefore, a transmission number that is unrealistically larger than $b_0$ can cause the model to be inconsistent. Moreover, values corresponding to realistic infectious disease parameters should lie in that range and yield consistency.

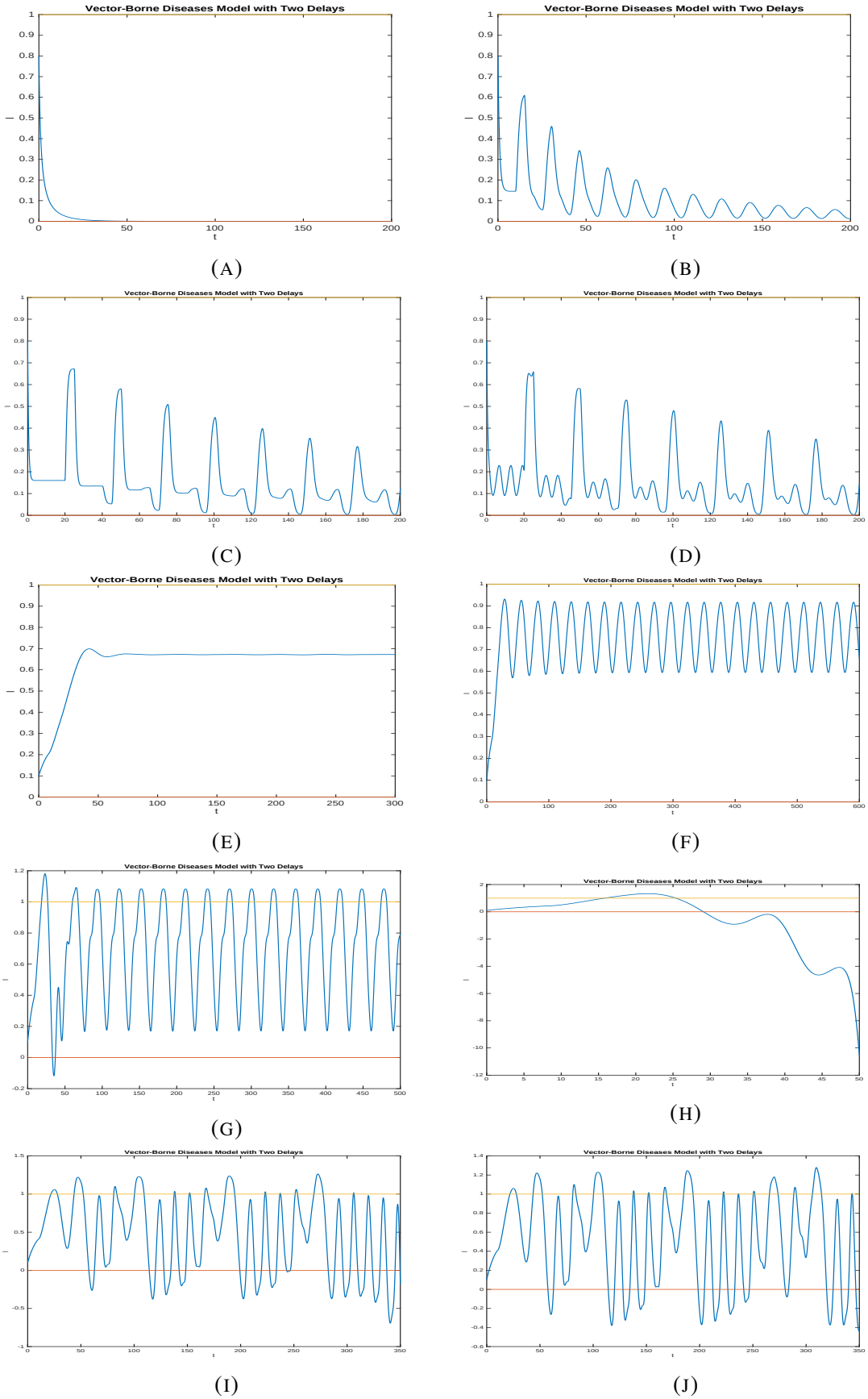


FIGURE 4.1. Numerical simulations of solutions to (1.3a)-(1.3b). See Table 4.1 for parameter values used in each simulation.

| Figures | b | c | e | $\tau_h$ | $\tau_v$ | tf | $\phi(t)$ | Note |
|---|---|---|---|---|---|---|---|---|
| Figure A | 1 | 1.1 | 1 | 0 | 0 | 200 | 0.8 | No delays, $c > b$, and $I_1^*$ is stable† |
| Figure B | 1 | 1.1 | 1 | 10 | 5 | 200 | 0.8 | With delays, $c > b$ and $I_1^*$ is stable |
| Figure C | 2 | 2 | 1 | 20 | 5 | 200 | 0.8 | $c = b$ and $I_1^*$ is stable |
| Figure D | 2 | 2 | 1 | 20 | 5 | 200 | $0.8 + 0.1 \sin t$ | Same as $B$ but with oscillatory $\phi$ |
| Figure E | 0.335 | 0.11 | 1 | 8.5 | 0 | 300 | 0.1 | $c < b < b_0$ |
| Figure F | 0.5 | 0.11 | 1 | 8.5 | 0 | 600 | 0.1 | Stable oscillations around $I_2^*$ for $b > b_0$ |
| Figure G | 0.69 | 0.11 | 1 | 8.5 | 0 | 500 | 0.1 | Solution crosses zero and one |
| Figure H | 0.8 | 0.11 | 1 | 8.5 | 0 | 50 | 0.1 | Solution falls down |
| Figure I | 0.69 | 0.11 | 1 | 8.5 | 2 | 350 | 0.1 | Before perturbation on $\phi$ |
| Figure J | 0.69 | 0.11 | 1 | 8.5 | 2 | 350 | 0.100000001 | After perturbation on $\phi$ |

TABLE 4.1. Parameter values used for each figure
†The zero equilibrium is denoted by $I_1^*$ and the endemic one is denoted by $I_2^*$

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```
function [ ] = VectorTwoDelaysFunction(b,c,e,delayHuman,delayVector,hist,tf)
% This function will solve the DDE modeling the spread of vector-borne
% diseases.  It will then plot the curve of the human infected propotion
% based on time.
% Function Inputs: b - Transmission number in model
%                  c - Recovery number in model
%                  e - e parameter in model
%                  delayHuman - Model's incubation period in humans
%                  delayVector - Model's incubation period in vectors
%                  tf - Stopping time
%                  hist - String that takes a valid history function of time
```

```
    % Time Step
    stept = tf*25;
    % Time-span vector
    tspan = [0 tf];

    % Checks if both delays are zero
    if (delayVector == 0) && (delayHuman == 0)
        % Evaluates the history function
        hist = @(t) eval(hist);
        % Gets the value of the history @ t = 0
        hist0 = hist(0);
        % Solves the ODE
        sol = ode45(@(t,u) b*u*(1 - e*u) - c*u, tspan,hist0);

    % Checks if the human delay is zero
    elseif delayHuman == 0
        % Delay variable
        lag = [delayVector];
        % Solves the DDE
        sol = dde23(@(t,u,Z) b*Z(1)*(1 - e*u) - c*u, lag,@(t) eval(hist),tspan);

    % Runs if the human delay is non-zero
    else
        % Delays row matrix
        lags = [delayHuman,delayHuman + delayVector];
        % Solves the DDE
        sol = dde23(@(t,u,Z) b*Z(2)*(1 - e*Z(1)) - c*u, lags,@(t) eval(hist),tspan);
    end

    % Stores the solution plot
    t = linspace(0,tf,stept);
    u = deval(sol, t);
    liney = zeros(1,length(t));
    liney1 = ones(1,length(t));

    % Draws the solution
```

```
    plot(t,u(1,:))
    hold('on')
    % Draws the zero axis
    plot(t, liney)
    % Draws the horizontal line of height one
    plot(t, liney1)
    hold('off')

    % Sets title and labels
    title("Vector-Borne Diseases Model with Two Delays")
    xlabel("t")
    ylabel("I")

end
```

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