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# Implementing lifetime performance index of products from type-II right - censored data using Lomax distribution

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ABSTRACT. Process capability analysis has been widely applied in the manufacturing industry to monitor the performance of industrial processes. The lifetime performance index $C_L$ is used to assess the performance and potential of their process, where $L$ is the lower specification limit. In the case of product processing a two parameter Lomax distribution, the study will apply the transformation technology to construct a maximum likelihood estimator (MLE) of $C_L$ based on type-II right-censored data. Then, the MLE of the $C_L$ is utilized to develop the new hypothesis testing procedure in the condition known as lower specification limit. Finally, we give an example and the Monte Carlo simulation to assess the behavior of the proposed method under the given significance level $\alpha$.

## 1. Introduction

In the manufacturing industry, quality of the products make an important contribution to long-term revenue and profitability. Quality is also able to change and maintain higher prices of products. For measuring the performance of a process, process capability indices (PCIs) are used to test the level of product quality. There are several PCIs in the literature (see Montgomery (1985)). Since capability is typically defined as the ability to carry out a task or achieve a goal, better process capability implies better product quality. A longer lifetime indicates better quality products.

The lifetime performance index, $C_L$ is used for evaluating the performance of the process. There are several studies that have been carried out in the literature addressing the topic of implementing $C_L$ under different distributions. Chen et al. (2002) developed the uniformly minimum variance unbiased estimator (UMVU) of electronic components lifetime performance index under an one parameter exponential distribution. The lifetime performance index was then utilized to develop the confidence interval. Tong et al. (2002) constructed a uniformly minimum variance unbiased estimator of electronic components lifetime performance index under an one parameter exponential distribution.Then the UMVU estimator of the lifetime performance index was utilized to develop the hypothesis testing procedure. Lee et al. (2009a) have developed a comprehensive study of the lifetime performance index of products with one parameter exponential distribution under progressively type II right censored data samples.

In the case of a product possessing a two-parameter exponential distribution, Lee et al. (2011a) constructed a UMVUE of the lifetime performance index based on the type-II right-censored sample. Then the UMVUE of the lifetime performance index was utilized to develop the new hypothesis testing procedure in the condition of known lower specification limit. Gunasekera and

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Wijekularathna (2019) constructed a generalized confidence limit for the performance index under two parameter exponential distribution based on type-II right censored data.

Many authors have studied the lifetime performance index for several distributions and the reader is referred to Lee et al. (2011b), Lee et al. (2009a), Lee et al. (2010), Dey et al. (2017), Hong and Wu (2017), Wu et al. (2014), Lee et al. (2009b), El-Sagheer (2017), Vishwakarma et al. (2018), Hong et al. (2009), Wijekularathna and Yi (2020), Wijekularathna and Subedi (2019).

The Lomax distribution is a very popular distribution in literature and often applied in the fields of business, actuarial science, economics, engineering and bio science. Some others have studied the estimation methods of parameters or confidence intervals for Lomax distribution. Al-Zahrani and Al-Sobhi (2013) derived the maximum likelihood estimator and Bayese stimators under the Lomax distribution based on general progressive censored data. Cramer and Schmiedt (2011) discussed the maximum likelihood estimates for the Lomax distribution. Moreover, the expected Fisher information matrix was computed. Okasha (2014) computed the estimators of unknown parameters in the Lomax distribution through the E-Bayesian method based on the type-II censoring schemes. Hu and Gui (2020) studied the lifetime performance index with Lomax distribution based on progressive type-I interval censored sample. They estimated the maximum likelihood estimator of $C_L$ and constructed a hypothesis test procedure when the scale parameter in the Lomax distribution is given.

In this article, we studied the lifetime performance index of Lomax distribution under type-II right censored data. Assuming the scale parameter is known, applying the data transformation techniques, the MLE of $C_L$ is constructed. The MLE of $C_L$ is then utilized to develop the new hypothesis testing procedure. The testing procedure can be employed to determine whether the lifetime of a product adheres to the required level.

## 2. The Lifetime Performance Index

Let $X$ be the lifetime of products, and $X$ has a Lomax distribution. $X$ will then have the probability density function (PDF)

$$f(x;\alpha,\theta) = \frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}},\ x>0,\ \theta>0,\ \alpha>0, \tag{2.1}$$

and the cumulative distribution function (CDF)

$$F(x;\alpha,\theta) = 1-\frac{\theta^{\alpha}}{(x+\theta)^{\alpha}},\ x>0,\ \theta>0,\ \alpha>0, \tag{2.2}$$

where $\alpha$ and $\theta$ are the scale and location parameters respectively.

The Lomax distribution can be converted to a one parameter exponential distribution by using the transformation $Y=\ln(1+\frac{X}{\theta})$ which has a PDF and CDF

$$f(y,\alpha) = \alpha e^{-\alpha y},\ y>0,\ \alpha>0, \tag{2.3}$$

and

$$F(y,\alpha) = 1-e^{-\alpha y},\ y>0,\ \alpha>0, \tag{2.4}$$

respectively.

It is well known that a longer lifetime implies a better product quality (Hong et al. (2012)). Hence, the lifetime is a larger-the-better type quality characteristic. Let $L$ be the lower specification limit for the lifetime of products. Montgomery (1985) defined a capability index $C_L$ by:

$$C_L = \frac{\mu - L}{\sigma},\ 0 \leq L < \infty, \tag{2.5}$$

where $\mu$ denotes the mean and $\sigma$ represents the standard deviation of the products. If the lifetime of a product exceeds the lowest specification limit (i.e. $X > L$), the product is labeled as a conforming product. This lifetime performance index can be used to assess the performance of the lifetime of products.

The failure rate function $h(t)$ is defined by

$$h(t) = \frac{f(t)}{1 - F(t)}, \tag{2.6}$$

where $f(t)$ and $F(t)$ are, respectively, the PDF and CDF of the products.

As we mentioned earlier, the random variable $Y$ has one-parameter exponential distribution. Thus, the lifetime performance index of Y, $C_{L_Y}$, can be rewritten as

$$C_{L_Y} = \frac{\mu_Y - L_X}{\sigma_Y} = \frac{\frac{1}{\alpha} - L_X}{\frac{1}{\alpha}} = 1 - \alpha L, \tag{2.7}$$

where

$$\mu_Y = E(Y) = \frac{1}{\alpha} \text{ and } \sigma_Y = \sqrt{Var(Y)} = \frac{1}{\alpha}.$$

The failure rate function of $Y$, *h(y)*, is defined by,

$$h(y) = \frac{f(y)}{1 - F(y)} = \alpha, \qquad \alpha > 0, \tag{2.8}$$

where $f(y)$ and $F(y)$ are the PDF and CDF given by equations (2.3) and (2.4) respectively.

Since the transformation $Y = \ln(1 + \frac{X}{\theta})$ is one-to-one and strictly increasing, the data set X and transformed data set Y have the same effect in assessing the performance index. When the mean $\frac{1}{\alpha} > L$, then the lifetime performance index $C_{L_Y} > 0$. From the above equations (2.7) and (2.8), it is clear that the larger the mean $\frac{1}{\alpha}$, the smaller the failure rate and the larger the lifetime performance index $C_L$. Therefore, the lifetime performance index, $C_{L_Y}$, reasonably and accurately represents the lifetime performance index of $X$, $C_{L_X}$. Then the conforming rate can be defined as

$$P_r = P(Y \geq L) = \int_L^\infty \alpha e^{-\alpha y} dy = e^{C_{L_Y} - 1},\ -\infty < C_{L_Y} \leq 1. \tag{2.9}$$

## 3. Maximum likelihood estimation of $C_{L_Y}$

A sample of $n$ items is selected at random from a population of Lomax distribution whose PDF and CDF are given by (2.1) and (2.2). The experiment is terminated as soon as the first $r$ units $X_1 < X_2 < ... < X_r$ fail. They would denote the $r$ smallest order statistics, a type-II censored data in a random sample of size $n$ from Lomax distribution. Then applying the transformation $Y = \ln(1+\frac{X}{\theta})$, we can have $r$ failure $Y_1 < Y_2 < ... < Y_r$ type-II censured data from one parameter exponential distribution where the PDF and CDF are given by (2.3) and (2.4), respectively. The joint PDF of type-II censored data is given by

$$L(y,\alpha) = \frac{n!}{(n-r)!}\left(\prod_{i=1}^{r} f(y_i)\right)[1-F(y_r)]^{n-r}\ , \tag{3.1}$$

where $f(y_i)$ and $F(y_i)$ are the PDF and CDF of Y given by (2.3) and (2.4), respectively.
Then, the likelihood function for $Y_1 < Y_2 < ... < Y_r$ is

$$L(y,\alpha) = c\left(\prod_{i=1}^{r} \alpha e^{-\alpha y_i}\right)(e^{-\alpha y_r})^{n-r}, \tag{3.2}$$

and the log-likelihood function is

$$l(y,\alpha) = \ln c + r\ln\alpha - \alpha(n-r)y_r - \alpha\sum_{i=1}^{r} y_i, \tag{3.3}$$

where $c = \frac{n!}{(n-r)!}$.
Hence, the MLE of $\alpha$ is given by

$$\frac{\partial l(y,\alpha)}{\partial\alpha} = \frac{r}{\alpha} - (n-r)y_r - \sum_{i=1}^{r} y_i = 0, \tag{3.4}$$

therefore,the MLE of $\alpha$, which is denoted by $\widehat{\alpha}$ can be given by

$$\widehat{\alpha} = \frac{r}{\sum_{i=1}^{r} y_i + (n-r)y_r},\ r \leq n. \tag{3.5}$$

By using the invariance property of MLE (see Zehna (1966)), the MLE of $C_{L_Y}$ can be given by

$$\widehat{C}_{L_Y} = 1 - \widehat{\alpha}L = 1 - \frac{rL}{\sum_{i=1}^{r} y_i + (n-r)y_r}. \tag{3.6}$$

Let $T = \sum_{i=1}^{r} y_i + (n-r)y_r$, then by using theorem 4.4.4 and Corollary 4.1.1 of Lawless (2003), $2T\alpha \sim \chi^2_{2r}$ where $\chi^2_{2r}$ denotes the chi-square distribution with degrees of freedom $2r$.
Then, the expectation of $\widehat{C}_{L_Y}$ can be obtained as follows:

$$\begin{aligned} E(\widehat{C}_{L_Y}) &= E\left(1 - \frac{rL}{\sum_{i=1}^{r} y_i + (n-r)y_r}\right) \\ &= 1 - 2\alpha rL\ E\left(\frac{1}{2\alpha T}\right) \\ &= 1 - \frac{\alpha rL}{r-1}. \end{aligned} \tag{3.7}$$

Since $E(\widehat{C}_{L_Y}) \neq C_L = 1 - \alpha L$, $\widehat{C}_{L_Y}$ is not an unbiased estimator of $C_{L_Y}$.
But consider the following;

$$
\begin{aligned}
E(\widehat{C}_{L_Y}) &= 1 - \frac{\alpha r L}{r-1} \\
&= 1 - \frac{\alpha L}{1 - 1/r} \\
&\overset{\text{when } r\to\infty}{=} 1 - \alpha L = C_{L_y}.
\end{aligned} \tag{3.8}
$$

Since $r$ approaches $\infty$, $E(\widehat{C}_{L_Y})$ approaches $C_{L_y}$, the MLE of $\widehat{C}_{L_Y}$ is an asymptotically unbiased estimator. Moreover,

$$
\begin{aligned}
Var(\widehat{C}_{L_Y}) &= Var\left(1 - \frac{rL}{\sum_{i=1}^{r} y_i + (n-r)y_r}\right) \\
&= 4\alpha^2 r^2 L^2 \, E\left(\frac{1}{2\alpha T}\right) \\
&= 4\alpha^2 r^2 L^2 \frac{2}{(2r-2)^2(2r-4)} \\
&= \frac{\alpha^2 r^2 L^2}{(r-1)^2(r-2)}.
\end{aligned} \tag{3.9}
$$

By Tchebysheff's theorem (see Wackerly et al. (2008) Theorem 4.13) and equation (3.9), we can write,

$$
\lim_{r\to\infty} P(|\widehat{C}_{L_Y} - C_L| > \epsilon) \le \lim_{r\to\infty} \frac{Var(\widehat{C}_{L_Y})}{\epsilon^2} = 0. \tag{3.10}
$$

Hence, by Wackerly et al. (2008), Theorem 9.1, we can conclude that $\widehat{C}_{L_Y}$ is a consistent estimator for $C_{L_Y}$.

## 4. Testing procedure for the lifetime performance index

In this section, we construct a statistical testing procedure to assess whether the lifetime performance index adheres to the required level. The process is capable if the lifetime performance index is larger than c*, where c* denotes the target value.

Then, the null and alternative hypothesis, $H_0$ and $H_1$ respectively can be presented as:

$$
H_0 : C_{L_Y} \le c^* \quad \text{vs} \quad H_1 : C_{L_Y} > c^*. \tag{4.1}
$$

Testing the above hypothesis with significance level $\gamma$ is equivalently finding the $100(1-\gamma)\%$ lower confidence bound for $C_{L_Y}$. By using $\widehat{C}_{L_Y}$, the MLE of $C_{L_Y}$ as the test statistic, given the specified significance level $\gamma$, the level $(1-\gamma)$ one-sided confidence interval for $C_{L_Y}$ can be derived as follows:

$$P(\widehat{C}_{L_Y} > C_0 | C_{L_Y} = c^*) = \gamma \tag{4.2}$$

$$P\left(1 - \frac{rL}{T} > C_0 | 1 - \alpha L = c^*\right) = \gamma \tag{4.3}$$

$$P\left(2\alpha T > \frac{2\alpha r L}{(1 - C_0)} | \alpha = \frac{1 - c^*}{L}\right) = \gamma \tag{4.4}$$

$$P\left(2\alpha T \le \frac{2r(1 - c^*)}{(1 - C_0)}\right) = 1 - \gamma, \tag{4.5}$$

where $2\alpha T \sim \chi^2_{2r}$.

From (4.5) the following can be obtained:

$$\frac{2r(1 - c^*)}{1 - C_0} = CHIINV(1 - \gamma, 2r), \tag{4.6}$$

where $CHIINV(1 - \gamma, 2r)$ is the lower (1-$\gamma$) percentile of $\chi^2_{2r}$.

Then, the critical value can be derived as

$$C_0 = 1 - \frac{2r(1 - c^*)}{CHIINV(1 - \gamma, 2r)}, \tag{4.7}$$

where $c^*, r,$ and $\gamma$ denote the target value, the specified significance level, and the observed number respectively. We can also see that $C_0$ is independent of $n$. Table 7.1 lists the critical values $C_0$ for $r = 2(1)50$ and c = 0.1(0.1)0.9 at $\gamma = 0.01$ and $\gamma = 0.05$.

Now, we will derive the lower confidence limit for the lifetime performance index $C_{L_Y}$.

Since $2\alpha T \sim \chi^2_{2r}$,

$$P(2\alpha T \le CHIINV(1 - \gamma, 2r)) = 1 - \gamma, \tag{4.8}$$

where $CHIINV(1 - \gamma, 2r)$ represents the lower $100(1 - \gamma)^{th}$ percentile of $\chi^2_{2r}$.

Therefore,

$$P\left(2\left(\frac{1 - C_{L_Y}}{L}\right)\left(\frac{rL}{1 - \widehat{C}_{L_Y}}\right) \le CHIINV(1 - \gamma, 2r)\right) = 1 - \gamma, \tag{4.9}$$

also,

$$P\left(C_{L_Y} \ge 1 - \frac{(1 - \widehat{C}_{L_Y})CHIINV(1 - \gamma, 2r)}{2r}\right) = 1 - \gamma. \tag{4.10}$$

Then,

$$C_{L_Y} \ge 1 - \frac{(1 - \widehat{C}_{L_Y})CHIINV(1 - \gamma, 2r)}{2r} \tag{4.11}$$

is the level 100(1-$\gamma$)% one-sided confidence interval for $C_{L_Y}$. Thus, the level 100(1-$\gamma$)% lower bound for $C_{L_Y}$ can be written as:

$$LB_{C_{L_Y}} = 1 - \frac{(1 - \widehat{C}_{L_Y})CHIINV(1 - \gamma, 2r)}{2r}, \tag{4.12}$$

where $\widehat{C}_{L_Y}$, $\gamma$, and $r$ denote the MLE of $C_{L_Y}$, the specified significance level, and the observed number, respectively.

The proposed testing procedure about $C_{L_Y}$ can be organized as follows:

**Algorithm 1:**

For given $(n, r, L, \alpha, \theta)$:

Step 1: Generate $x_i$ from Lomax distribution with PDF given by Equation (2.1) for $i = 1, 2, ..., n$ with the aid of $X = \theta[(1-u)^{\frac{-1}{\alpha}} - 1]$, where $u \sim u(0,1)$.

Step 2: Let the data transformation $y_i = \ln(1 + \frac{x_i}{\theta}), i = 1, 2, ..., n$ for the type-II right censured data $x_1, x_2, ..., x_n$.

Step 3: Compute the MLE of $\alpha$, $\widehat{\alpha}_{\mu L}$ and estimate of $C_{L_Y}$, $\widehat{C}_{L_Y}$ using

$$\widehat{C}_{L_Y} = 1 - \widehat{\alpha}L. \tag{4.13}$$

Step 4: Calculate the 100(1-$\gamma$)% lower bound, $LB_{C_{L_Y}}$ for $C_{L_Y}$ from (4.12).

Step 5: If the true value of the life performance index $C_{L_Y}$ is greater than the lower bound, i.e., $LB_{C_{L_Y}} < C_{L_Y}$, the counting amount COUNT = 1, else COUNT = 0.

Step 6: Repeat Step 1 through Step 5 10,000 times.

Step 7: The coverage probability is the average of COUNT = 1.

The proposed testing procedure about the performance index $C_{L_Y}$ can be presented as follows:

**Algorithm 2:**

For given ($L$, $r$, c*, $\gamma$):

For given the lower lifetime limit L for products and performance index value $c*$, the null $(H_0)$ and alternative $(H_1)$ hypothesis can be presented as follows:

$$H_0 : C_{L_Y} \leq c^* \text{ vs } H_1 : C_{L_Y} > c^*. \tag{4.14}$$

Step 1: Generate $x_i$ from Lomax distribution with PDF given by Equation (2.1) for $i = 1, 2, ..., n$ with the aid of $X = \theta[(1-u)^{\frac{-1}{\alpha}} - 1]$, where $u \sim u(0,1)$.

Step 2: Let the data transformation $y_i = \ln(1 + \frac{x_i}{\theta})$; $i = 1, 2, ..., n$ for the type-II right censured data $x_1, x_2, ..., x_n$.

Step 3: Calculate the value of the test statistic

$$\widehat{C}_{L_Y} = 1 - \frac{rL}{\sum_{i=1}^{r} y_i + (n-r)y_r}. \tag{4.15}$$

Step 4: Obtain the critical value $C_0$ from

$$C_0 = 1 - \frac{2r(1-c^*)}{CHIINV(1-\gamma, 2r)}, \tag{4.16}$$

for the given significance level $\gamma$, the number of observed failures $r$ and the target value c*.

Step 5: The decision value of the statistical test can be provided as follows: If $\widehat{C}_{L_Y} > C_0$, it is concluded that the lifetime performance index of product meets the required level.

## 5. The Monte Carlo simulation of power

Consider the following null ($H_0$) and alternative ($H_1$) hypothesis for the given target value c*.

$$H_0 : C_{L_Y} \leq c* \ \text{ vs } \ H_1 : C_{L_Y} > c* \, . \tag{5.1}$$

Then the power of statistical test can be derived as follows:

Under the type-II right censoring scheme, we obtain a size $\gamma$ test with the rejection region $[\widehat{C}_{L_Y} > C_0]$, for the observed censored site and sample size $n$ $(r \leq n)$. The power, $P(C_1)$ of the test at the point $C_L = C_1(C > c)$ can be derived as follows

$$\begin{aligned}
P(C_1) &= P\left(\widehat{C}_{L_Y} > C_0 | C_{L_Y} = C_1\right), \\
&= P\left(1 - \frac{rL}{T} > 1 - \frac{2r(1-c^*)}{CHIINV(1-\gamma, 2r)} | 1 - \alpha L = C_1\right), \\
&= P\left(2\alpha T > \frac{CHIINV(1-\gamma, 2r)L\alpha}{(1-c^*)} | \alpha = \frac{1-C_1}{L}\right), \\
&= P\left(2\alpha T > \frac{(1-C_1)CHIINV(1-\gamma, 2r)}{1-c^*}\right)
\end{aligned} \tag{5.2}$$

where $2\alpha T \sim \chi^2_{2r}$.

Procedure of the Monte Carlo Simulation of power function can be represented as follows:

**Algorithm 3:**

For given $C,\ c*,\ L,\ \gamma,\ r,\ n$ where $c^* < C_1 < 1$ and $\gamma \leq n$

Step 1: a) The value of $\alpha$ is calculated to be the equation (2.7) as follows:

$$C_{L_Y} = 1 - \alpha L = C_1;\ C_{L_Y} < 1, \tag{5.3}$$

$$\alpha = \frac{1 - C_1}{L}. \tag{5.4}$$

b) Generate $x_i$ from Lomax distribution with PDF given by Equation (2.1) for $i = 1, 2, ..., n$ with the aid of $X = \theta[(1-u)^{\frac{-1}{\alpha}} - 1]$, where $u \sim U(0, 1)$.

c) Let the data transformation $y_i = \ln(1 + \frac{x_i}{\theta})$; $i = 1, 2, ..., n$ for the type-II right censured data $x_1, x_2, ..., x_n$.

d) The ranking of random samples is $Y_1 < Y_2 < \cdots < Y_n$ and the first r observation $Y_1 < Y_2 < \cdots < Y_r$ are used.

e) The value of $\widehat{C}_{L_Y}$ is calculated by

$$\widehat{C}_{L_Y} = 1 - \frac{rL}{T} \text{ where } T = \sum_{i=1}^{r} \ln(y_i) + (n - r). \tag{5.5}$$

f) If $\widehat{C}_{L_Y} > C_0$ then COUNT = 1, else COUNT = 0, where $C_0 = 1 - \frac{2r(1-c^*)}{CHIINV(1-\gamma,2r)}$.

Step 2: a) Step 1 is repeated 1,000 times.

b) The estimation of power $P(C_1)$ is

$$\widehat{P}(C_1) = \frac{\text{Total } COUNT}{1,000}. \tag{5.6}$$

Based on 100 estimations of power $\widehat{P}_1(C_1),\ \widehat{P}_2(C_2),\ldots,\widehat{P}_{100}(C_{100})$ the sample mean square error (SMSE) can be computed as

$$SMSE = \frac{\sum_{i=1}^{100}(\widehat{P}_i(C_i) - P(C_1))^2}{100}, \tag{5.7}$$

where $P(C_1)$ is given by (5.2). The simulation power is given by

$$\overline{\widehat{P}(C_1)} = \frac{\sum_{i=1}^{100}\widehat{P}_i(C_i)}{100}. \tag{5.8}$$

The following Tables 5.1 - 5.9 depict the results of the power simulation. They compare the real power with simulation power and calculated SMSE.

TABLE 5.1. The values of $P(c_1),\ \overline{\widehat{P}(c_1)}$ and SMSE for $n = 10, r = 3$, and $\alpha = 0.05[0.01]$.

| $c_1$ | $P(c_1)$ | $\overline{\widehat{P(c_1)}}$ | SMSE |
|---|---|---|---|
| 0.1 | 0.05000 [0.01000] | 0.05013 [0.01295] | 4.90e-07 [9.23e-06] |
| 0.2 | 0.08261 [0.02070] | 0.09400 [0.02895] | 0.00013 [6.86e-05] |
| 0.3 | 0.13362 [0.04185] | 0.14253 [0.04482] | 8.06e-05 [9.39e-06] |
| 0.4 | 0.21061 [0.08216] | 0.21391 [0.09220] | 1.28e-05 [0.000101] |
| 0.5 | 0.32128 [0.15534] | 0.33581 [0.16344] | 0.000215 [6.92e-05] |
| 0.6 | 0.46990 [0.27939] | 0.46761 [0.28812] | 9.24e-06 [7.92e-05] |
| 0.7 | 0.65001 [0.46898] | 0.63590 [0.46702] | 0.000204 [9.39e-06] |
| 0.8 | 0.83372 [0.71235] | 0.81737 [0.70837] | 0.000284 [2.01e-05] |
| 0.9 | 0.96592 [0.93143] | 0.96728 [0.92635] | 3.10e-06 [3.26e-05] |

TABLE 5.2. The values of $P(c_1),\ \overline{\widehat{P}(c_1)}$ and SMSE for $n = 10, r = 5$, and $\alpha = 0.05[0.01]$

| $c_1$ | $P(c_1)$ | $\overline{\widehat{P(c_1)}}$ | SMSE |
|---|---|---|---|
| 0.1 | 0.05000 [0.01000] | 0.05959 [0.01242] | 9.22e-05 [6.10e-06] |
| 0.2 | 0.09208 [0.02382] | 0.10836 [0.03080] | 0.000266 [4.90e-05] |
| 0.3 | 0.16237 [0.05410] | 0.17563 [0.06432] | 0.000178 [0.000105] |
| 0.4 | 0.27159 [0.11575] | 0.26934 [0.12830] | 7.11e-06 [0.000160] |
| 0.5 | 0.42566 [0.22966] | 0.41338 [0.22806] | 0.000155 [6.15e-06] |
| 0.6 | 0.61551 [0.41329] | 0.60304 [0.40138] | 0.000182 [0.000146] |
| 0.7 | 0.80659 [0.65457] | 0.81310 [0.64005] | 4.87e-05 [0.000226] |
| 0.8 | 0.94422 [0.88040] | 0.94334 [0.87631] | 5.99e-06 [2.10e-05] |
| 0.9 | 0.99607 [0.98968] | 0.99602 [0.98640] | 5.22e-07 [1.10e-05] |

TABLE 5.3. The values of $P(c_1)$, $\overline{\widehat{P}(c_1)}$ and SMSE for$n = 10, r = 10$, and $\alpha = 0.05[0.01]$.

| $c_1$ | $P(c_1)$ | $\overline{\widehat{P(c_1)}}$ | SMSE |
|---|---|---|---|
| 0.1 | 0.05000 [0.01000] | 0.04891 [0.01243] | 1.45e-06 [6.85e-06] |
| 0.2 | 0.11130 [0.03054] | 0.12654 [0.03256] | 0.000234 [4.46e-06] |
| 0.3 | 0.22410 [0.08355] | 0.24681 [0.09617] | 0.000519 [0.00016] |
| 0.4 | 0.40065 [0.19975] | 0.40682 [0.21862] | 4.69e-05 [0.000363] |
| 0.5 | 0.62357 [0.40481] | 0.61966 [0.41182] | 2.76e-05 [5.80e-05] |
| 0.6 | 0.83251 [0.67261] | 0.82458 [0.66170] | 6.84e-05 [0.000131] |
| 0.7 | 0.95882 [0.89694] | 0.95650 [0.89591] | 6.55e-06 [4.55e-06] |
| 0.8 | 0.99675 [0.98930] | 0.99379 [0.98198] | 8.93e-06 [5.40e-05] |
| 0.9 | 0.99999 [0.99993] | 1.00000 [1.00000] | 2.23e-10 [4.34e-09] |

TABLE 5.4. The values of $P(c_1)$, $\overline{\widehat{P}(c_1)}$ and SMSE for $n = 20, r = 10$, and $\alpha = 0.05[0.01]$.

| $c_1$ | $P(c_1)$ | $\overline{\widehat{P(c_1)}}$ | SMSE |
|---|---|---|---|
| 0.1 | 0.05000 [0.01000] | 0.05993 [0.00998] | 9.91e-05 [2.00e-08] |
| 0.2 | 0.11130 [0.03054] | 0.11646 [0.04100] | 2.76e-05 [0.00011] |
| 0.3 | 0.22410 [0.08355] | 0.22302 [0.09374] | 9.02e-06 [0.000104] |
| 0.4 | 0.40065 [0.19975] | 0.40595 [0.20584] | 2.93e-05 [4.35e-05] |
| 0.5 | 0.62357 [0.40481] | 0.62639 [0.41095] | 1.36e-05 [3.89e-05] |
| 0.6 | 0.83251 [0.67261] | 0.83882 [0.67091] | 4.33e-05 [8.67e-06] |
| 0.7 | 0.95882 [0.89694] | 0.95861 [0.90141] | 8.44e-07 [2.29e-05] |
| 0.8 | 0.99675 [0.98930] | 0.99728 [0.98928] | 4.83e-07 [2.02e-07] |
| 0.9 | 0.99999 [0.99993] | 1.00000 [1.00000] | 2.23e-10 [4.34e-09] |

TABLE 5.5. The values of $P(c_1)$, $\overline{\widehat{P}(c_1)}$ and SMSE for $n = 20, r = 15$, and $\alpha = 0.05[0.01]$.

| $c_1$ | $P(c_1)$ | $\overline{\widehat{P(c_1)}}$ | SMSE |
|---|---|---|---|
| 0.1 | 0.05000 [0.01000] | 0.06402 [0.00849] | 0.000198 [2.53e-06] |
| 0.2 | 0.12777 [0.03668] | 0.14777 [0.05139] | 0.000401 [0.000217] |
| 0.3 | 0.27901 [0.11320] | 0.27413 [0.13286] | 2.57e-05 [0.000388] |
| 0.4 | 0.50805 [0.28372] | 0.51715 [0.27513] | 8.58e-05 [7.56e-05] |
| 0.5 | 0.75744 [0.55595] | 0.76331 [0.55851] | 3.97e-05 [1.25e-05] |
| 0.6 | 0.92997 [0.83060] | 0.91383 [0.83527] | 0.000262 [2.43e-05] |
| 0.7 | 0.99185 [0.97300] | 0.98947 [0.96666] | 5.93e-06 [4.10e-05] |
| 0.8 | 0.99983 [0.99921] | 1.00000 [0.99700] | 2.87e-08 [4.89e-06] |
| 0.9 | 1.00000 [1.00000] | 1.00000 [1.00000] | 2.37e-15 [1.05e-13] |

TABLE 5.6. The values of $P(c_1)$, $\overline{\widehat{P}(c_1)}$ and SMSE for $n = 20, r = 20$, and $\alpha = 0.05[0.01]$

| $c_1$ | $P(c_1)$ | $\overline{\widehat{P}(c_1)}$ | SMSE |
|---|---|---|---|
| 0.1 | 0.05000 [0.01000] | 0.06147 [0.01271] | 0.000132 [7.55e-06] |
| 0.2 | 0.14289 [0.04260] | 0.14779 [0.04808] | 2.63e-05 [3.07e-05] |
| 0.3 | 0.32977 [0.14345] | 0.32840 [0.14879] | 5.98e-06 [3.07e-05] |
| 0.4 | 0.59826 [0.36546] | 0.59415 [0.35865] | 3.81e-05 [5.10e-05] |
| 0.5 | 0.84623 [0.67798] | 0.82668 [0.67084] | 0.000385 [6.19e-05] |
| 0.6 | 0.97166 [0.91705] | 0.96811 [0.91034] | 1.29e-05 [4.68e-05] |
| 0.7 | 0.99847 [0.99352] | 0.99757 [0.98857] | 1.06e-06 [2.47e-05] |
| 0.8 | 0.99999 [0.99995] | 1.00000 [1.00000] | 6.71e-11 [2.64e-09] |
| 0.9 | 1.00000 [1.00000] | 1.00000 [1.00000] | 2.07e-20 [1.84e-18] |

TABLE 5.7. The values of $P(c_1)$, $\overline{\widehat{P}(c_1)}$ and SMSE for $n = 30, r = 10$, and $\alpha = 0.05[0.01]$.

| $c_1$ | $P(c_1)$ | $\overline{\widehat{P}(c_1)}$ | SMSE |
|---|---|---|---|
| 0.1 | 0.05000 [0.01000] | 0.04850 [0.00400] | 3.98e-06 [3.66e-05] |
| 0.2 | 0.11130 [0.03054] | 0.11495 [0.02613] | 1.66e-05 [2.04e-05] |
| 0.3 | 0.22410 [0.08355] | 0.22506 [0.08671] | 8.14e-06 [1.05e-05] |
| 0.4 | 0.40065 [0.19975] | 0.37873 [0.20161] | 0.000490 [8.79e-06] |
| 0.5 | 0.62357 [0.40481] | 0.59911 [0.38617] | 0.000619 [0.000359] |
| 0.6 | 0.83251 [0.67261] | 0.80960 [0.65130] | 0.000529 [0.000471] |
| 0.7 | 0.95882 [0.89694] | 0.94937 [0.87340] | 8.97e-05 [0.000556] |
| 0.8 | 0.99675 [0.98930] | 1.00000 [0.99504] | 1.06e-05 [3.32e-05] |
| 0.9 | 0.99999 [0.99993] | 1.00000 [1.00000] | 2.23e-10 [4.34e-09] |

TABLE 5.8. The values of $P(c_1)$, $\overline{\widehat{P}(c_1)}$ and SMSE for $n = 30, r = 20$, and $\alpha = 0.05[0.01]$.

| $c_1$ | $P(c_1)$ | $\overline{\widehat{P}(c_1)}$ | SMSE |
|---|---|---|---|
| 0.1 | 0.05000 [0.01000] | 0.04467 [0.00671] | 2.86e-05 [1.10e-05] |
| 0.2 | 0.14289 [0.04260] | 0.13304 [0.03367] | 9.98e-05 [8.00e-05] |
| 0.3 | 0.32977 [0.14345] | 0.33211 [0.13404] | 2.02e-05 [9.15e-05] |
| 0.4 | 0.59826 [0.36546] | 0.59198 [0.35940] | 5.33e-05 [5.52e-05] |
| 0.5 | 0.84623 [0.67798] | 0.83548 [0.66819] | 0.000121 [0.000112] |
| 0.6 | 0.97166 [0.91705] | 0.96333 [0.90942] | 7.05e-05 [6.63e-05] |
| 0.7 | 0.99847 [0.99352] | 0.99700 [0.99190] | 2.16e-06 [2.71e-06] |
| 0.8 | 0.99999 [0.99995] | 1.00000 [1.00000] | 6.71e-11 [2.64e-09] |
| 0.9 | 1.00000 [1.00000] | 1.00000 [1.00000] | 2.07e-20 [1.84e-18] |

TABLE 5.9. The values of $P(c_1)$, $\overline{\widehat{P}(c_1)}$ and SMSE for $n = 30, r = 30$, and $\alpha = 0.05[0.01]$.

| $c_1$ | $P(c_1)$ | $\overline{\widehat{P(c_1)}}$ | SMSE |
|---|---|---|---|
| 0.1 | 0.05000 [0.01000] | 0.04340 [0.01100] | 4.44e-05 [1.00e-06] |
| 0.2 | 0.17082 [0.05424] | 0.17495 [0.04734] | 2.23e-05 [4.82e-05] |
| 0.3 | 0.42179 [0.20546] | 0.41136 [0.20098] | 0.000142 [2.59e-05] |
| 0.4 | 0.73626 [0.51525] | 0.73117 [0.50183] | 5.38e-05 [0.000233] |
| 0.5 | 0.94066 [0.84153] | 0.93386 [0.82891] | 4.68e-05 [0.000179] |
| 0.6 | 0.99569 [0.98234] | 0.99057 [0.97765] | 2.64e-05 [2.28e-05] |
| 0.7 | 0.99995 [0.99969] | 1.00000 [1.00000] | 2.33e-09 [9.80e-08] |
| 0.8 | 1.00000 [1.00000] | 1.00000 [1.00000] | 2.73e-16 [2.99e-14] |
| 0.9 | 1.00000 [1.00000] | 1.00000 [1.00000] | 9.98e-31 [3.16e-28] |

## 6. Numerical example

In this section we give a numerical example to illustrate the methods of inference developed in the preceding sections. We have generated an artificial type-II censored sample of size $n = 25$ from a Lomax distribution with the PDF defined by (2.1). Here, the data generated form the parameters $\sigma$ = 1.5 and $\alpha$ = 2. This data is given in the Table 6.1.

TABLE 6.1. The artificial type-II censored sample of size n = 25 with the parameters $\sigma$ = 1.5 and $\alpha$ = 2.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0.085 | 0.090 | 0.177 | 0.179 | 0.189 | 0.205 | 0.247 | 0.260 |
| 0.273 | 0.317 | 0.417 | 0.601 | 0.683 | 0.748 | 0.759 | 1.174 |
| 1.243 | 1.338 | 1.923 | 2.276 | 2.980 | 3.027 | 3.703 | 5.692 |

In order to estimate the parameter $\theta$ of the Lomax distribution, we define the Gini statistics (Gail and Gastwirth, 1978), as suggested by Lee et al. (2011a).

$$G_n = \frac{\sum_{i=1}^{n-1} iD_{i+1}}{(n-1)\sum_{i=1}^{n} D_i}, \tag{6.1}$$

where $D_i = (n - i + 1)(Y_{(i)} - Y_{(i+1)})$ for $i = 1, 2, ..., n$ and $D_i = ny$, while $Y_1 = \ln(1 + \frac{x_1}{\theta})$.

TABLE 6.2. P-values for corresponding $\theta$ values.

| $\theta$ | p-value | $\theta$ | p-value | $\theta$ | p-value | $\theta$ | p-value |
|---|---|---|---|---|---|---|---|
| 1.30 | 0.8741781 | 1.48 | 0.9835675 | 1.66 | 0.9229449 | 1.84 | 0.8429334 |
| 1.31 | 0.8807139 | 1.49 | 0.9891556 | 1.67 | 0.9181679 | 1.85 | 0.8388378 |
| 1.32 | 0.8871937 | 1.50 | 0.9943704 | 1.68 | 0.9134320 | 1.86 | 0.8347765 |
| 1.33 | 0.8936178 | **1.51** | **0.9998138** | 1.69 | 0.9087369 | 1.87 | 0.8307493 |
| 1.34 | 0.8999865 | 1.52 | 0.9943704 | 1.70 | 0.9040821 | 1.88 | 0.8267558 |
| 1.35 | 0.9063004 | 1.53 | 0.9889744 | 1.71 | 0.8994672 | 1.89 | 0.8227956 |
| 1.36 | 0.9125597 | 1.54 | 0.9836253 | 1.72 | 0.8948918 | 1.90 | 0.8188684 |
| 1.37 | 0.9187649 | 1.55 | 0.9783228 | 1.73 | 0.8903557 | 1.91 | 0.8149738 |
| 1.38 | 0.9249163 | 1.56 | 0.9730664 | 1.74 | 0.8858582 | 1.92 | 0.8111117 |
| 1.39 | 0.9310145 | 1.57 | 0.9678556 | 1.75 | 0.8813992 | 1.93 | 0.8072815 |
| 1.40 | 0.9370597 | 1.58 | 0.9626901 | 1.76 | 0.8769782 | 1.94 | 0.8034830 |
| 1.41 | 0.9430525 | 1.59 | 0.9575694 | 1.77 | 0.8725948 | 1.95 | 0.7997159 |
| 1.42 | 0.9489932 | 1.60 | 0.9524931 | 1.78 | 0.8682486 | 1.96 | 0.7959799 |
| 1.43 | 0.9548822 | 1.61 | 0.9474609 | 1.79 | 0.8639393 | 1.97 | 0.7922746 |
| 1.44 | 0.9607200 | 1.62 | 0.9424722 | 1.80 | 0.8596666 | 1.98 | 0.7885997 |
| 1.45 | 0.9665070 | 1.63 | 0.9375266 | 1.81 | 0.8554300 | 1.99 | 0.7849549 |
| 1.46 | 0.9722437 | 1.64 | 0.9326238 | 1.82 | 0.8512291 | 2.00 | 0.7813400 |
| 1.47 | 0.9779304 | 1.65 | 0.9277634 | 1.83 | 0.8470637 | 2.01 | 0.7777545 |

For $n > 20$, the statistic $(12(n-1))^{1/2}(G_n - 0.5)$ tends to the standard normal distribution, $N(0,1)$. Hence the p-value = $P\{|z| > |(12(n-1))^{1/2}(g_n - 0.5)|\}$, where $g_n$ is the observed value of $G_n$ and z has an approximate of $N(0,1)$. So, by using the maximum p-value method, the optimum value of $\theta$ is selected and then we suppose $\theta$ is known. For the data set in Table 6.1, the values of $\theta$ and the corresponding $p$-values are shown in Table 6.2. Table 6.2 indicates that $\theta$ = 1.51 is very close to the optimum value and the maximum p-value = 0.9998138. So, we assume that the data set has Lomax distribution with the following PDF:

$$f(x;\alpha,1.51) = \frac{\alpha 1.51^{\alpha}}{(x+1.51)^{\alpha}},\ x > 0. \tag{6.2}$$

The proposed testing procedure about $C_{L_X}$ can be stated as follows. From the sample, we randomly select $q$ observations, i.e. $r = q$, for our testing procedure. Then, the transformation $Y = \ln(1 + \frac{x}{1.51})$ is applied to the $q$ observation. The selected data and the transformed data are presented in Table 6.3.

TABLE 6.3. Selected and transformed data

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $X_i$ | 0.090 | 0.177 | 0.205 | 0.317 | 0.417 | 0.748 | 1.243 | 1.923 | 5.692 |
| $Y_i$ | 0.058 | 0.111 | 0.127 | 0.191 | 0.244 | 0.402 | 0.601 | 0.821 | 1.562 |

The subsequent steps of the analysis are given below.

Step 1: We assume that the lower lifetime limit $L_X$ is 0.105. The conforming rate $P_r$ of the product is assumed to exceed 90%. From equation (2.7), we find that that the $C_{L_X}$ value will exceed 0.9. Thus the performance index, c*, will be assumed to be 0.9, i.e. c*=0.9. Then, the null and alternative hypothesis can be written as:

$$H_0 : C_{L_Y} \leq 0.9 \text{ vs } H_1 : C_{L_Y} > 0.9. \tag{6.3}$$

Step 2: Specify the significance level of $\gamma = 0.05$. By the equation (4.5), the critical value $C_0$ can be calculated as $C_0 = 0.93765$.

Step 3: Calculate the test statistic,

$$\begin{aligned} \widehat{C}_{L_Y} &= 1 - \frac{rL}{\sum_{i=1}^{\gamma} y_i + (n-r)y_r} \\ &= 0.96754. \end{aligned} \tag{6.4}$$

Step 4: Since $\widehat{C}_{L_Y} = 0.96754 > C_0 = 0.93765$, the null hypothesis $H_0 : C_{L_Y} \leq 0.9$ is rejected in favor of $H_1 : C_{L_Y} > 0.9$. Thus, the conclusion is that the lifetime performance index of the product meets the required level.

From (4.12), the 95% lower bound of $C_{L_Y}$ is obtained by

$$LB_{C_{L_Y}} = 1 - \frac{\chi^2_{1-\gamma}, 18}{2r}(1 - \widehat{C}_{L_Y}), \tag{6.5}$$

with $\chi^2_{0.95,18} = 28.8693$. Then the lower bound, $LB_{C_{L_Y}}$ can be calculated at $LB_{C_{L_Y}} = 0.94793$. Since c* = 0.9 $\notin [0.94793, \infty)$, we reject the null hypothesis $H_0 : C_{L_Y} \leq 0.9$ in favor of $H_1 : C_{L_Y} > 0.78$. Thus, the conclusion is that the lifetime performance index of a product meets the required level.

## 7. Conclusion

Process capability analysis has been widely applied in the manufacturing industry to monitor the performance of industrial processes. Several authors have developed different statistical procedures to assess the lifetime performance of products under different distributions. This study is conducted with Lomax distributed products. The Lomax distribution is very suitable to describe the characteristic of lifetime. This research constructs methods of assessing the lifetime performance index $C_L$ of products with the Lomax distribution under type-II censored samples. The MLE of $C_L$ is inferred by data transformation and then utilized to develop a hypothesis testing procedure and a confidence interval to assess product performance. The Monte Carlo simulation results and numerical example results show that our proposed method exhibits good properties. In summary, the proposed testing procedure can effectively evaluate whether the lifetime of products meets the requirement.

TABLE 7.1. Critical Values ($C_0$) for given $c$ and $r$ values.

| r | c=0.1 | c=0.2 | c=0.3 | c=0.4 | c=0.5 |
|---:|---:|---:|---:|---:|---:|
| 1.0000 | 0.6996 [0.8046] | 0.7330 [0.8263] | 0.7663 [0.8480] | 0.7997 [0.8697] | 0.8331 [0.8914] |
| 2.0000 | 0.6206 [0.7288] | 0.6627 [0.7590] | 0.7049 [0.7891] | 0.7470 [0.8192] | 0.7892 [0.8494] |
| 3.0000 | 0.5711 [0.6788] | 0.6188 [0.7145] | 0.6664 [0.7502] | 0.7141 [0.7859] | 0.7617 [0.8216] |
| 4.0000 | 0.5357 [0.6416] | 0.5873 [0.6814] | 0.6389 [0.7213] | 0.6905 [0.7611] | 0.7421 [0.8009] |
| 5.0000 | 0.5084 [0.6122] | 0.5630 [0.6553] | 0.6176 [0.6984] | 0.6723 [0.7415] | 0.7269 [0.7846] |
| 6.0000 | 0.4864 [0.5881] | 0.5434 [0.6338] | 0.6005 [0.6796] | 0.6576 [0.7254] | 0.7146 [0.7711] |
| 7.0000 | 0.4680 [0.5676] | 0.5271 [0.6157] | 0.5862 [0.6637] | 0.6453 [0.7117] | 0.7045 [0.7598] |
| 8.0000 | 0.4524 [0.5500] | 0.5132 [0.6000] | 0.5741 [0.6500] | 0.6349 [0.7000] | 0.6958 [0.7500] |
| 9.0000 | 0.4389 [0.5346] | 0.5012 [0.5863] | 0.5636 [0.6380] | 0.6259 [0.6897] | 0.6883 [0.7414] |
| 10.0000 | 0.4269 [0.5208] | 0.4906 [0.5741] | 0.5543 [0.6273] | 0.6180 [0.6806] | 0.6816 [0.7338] |
| 11.0000 | 0.4163 [0.5086] | 0.4812 [0.5632] | 0.5460 [0.6178] | 0.6109 [0.6724] | 0.6757 [0.7270] |
| 12.0000 | 0.4068 [0.4974] | 0.4727 [0.5533] | 0.5387 [0.6091] | 0.6046 [0.6650] | 0.6705 [0.7208] |
| 13.0000 | 0.3982 [0.4873] | 0.4651 [0.5443] | 0.5320 [0.6012] | 0.5988 [0.6582] | 0.6657 [0.7152] |
| 14.0000 | 0.3904 [0.4780] | 0.4581 [0.5360] | 0.5259 [0.5940] | 0.5936 [0.6520] | 0.6613 [0.7100] |
| 15.0000 | 0.3832 [0.4695] | 0.4517 [0.5284] | 0.5203 [0.5874] | 0.5888 [0.6463] | 0.6573 [0.7053] |
| 16.0000 | 0.3765 [0.4615] | 0.4458 [0.5214] | 0.5151 [0.5812] | 0.5844 [0.6410] | 0.6536 [0.7009] |
| 17.0000 | 0.3704 [0.4542] | 0.4404 [0.5148] | 0.5103 [0.5755] | 0.5803 [0.6361] | 0.6502 [0.6968] |
| 18.0000 | 0.3647 [0.4473] | 0.4353 [0.5087] | 0.5059 [0.5701] | 0.5765 [0.6315] | 0.6470 [0.6929] |
| 19.0000 | 0.3594 [0.4408] | 0.4305 [0.5030] | 0.5017 [0.5651] | 0.5729 [0.6272] | 0.6441 [0.6894] |
| 20.0000 | 0.3544 [0.4348] | 0.4261 [0.4976] | 0.4978 [0.5604] | 0.5696 [0.6232] | 0.6413 [0.6860] |
| 21.0000 | 0.3497 [0.4291] | 0.4219 [0.4925] | 0.4942 [0.5559] | 0.5664 [0.6194] | 0.6387 [0.6828] |
| 22.0000 | 0.3452 [0.4237] | 0.4180 [0.4877] | 0.4907 [0.5517] | 0.5635 [0.6158] | 0.6362 [0.6798] |
| 23.0000 | 0.3411 [0.4186] | 0.4143 [0.4832] | 0.4875 [0.5478] | 0.5607 [0.6124] | 0.6339 [0.6770] |
| 24.0000 | 0.3371 [0.4137] | 0.4108 [0.4788] | 0.4844 [0.5440] | 0.5581 [0.6091] | 0.6317 [0.6743] |
| 25.0000 | 0.3334 [0.4091] | 0.4074 [0.4747] | 0.4815 [0.5404] | 0.5556 [0.6061] | 0.6297 [0.6717] |
| 26.0000 | 0.3298 [0.4047] | 0.4043 [0.4708] | 0.4788 [0.5370] | 0.5532 [0.6031] | 0.6277 [0.6693] |
| 27.0000 | 0.3264 [0.4005] | 0.4013 [0.4671] | 0.4761 [0.5337] | 0.5510 [0.6003] | 0.6258 [0.6669] |
| 28.0000 | 0.3232 [0.3965] | 0.3984 [0.4636] | 0.4736 [0.5306] | 0.5488 [0.5977] | 0.6240 [0.6647] |
| 29.0000 | 0.3201 [0.3927] | 0.3957 [0.4602] | 0.4712 [0.5276] | 0.5467 [0.5951] | 0.6223 [0.6626] |
| 30.0000 | 0.3172 [0.3890] | 0.3930 [0.4569] | 0.4689 [0.5248] | 0.5448 [0.5927] | 0.6206 [0.6606] |
| 31.0000 | 0.3143 [0.3855] | 0.3905 [0.4538] | 0.4667 [0.5220] | 0.5429 [0.5903] | 0.6191 [0.6586] |
| 32.0000 | 0.3116 [0.3821] | 0.3881 [0.4507] | 0.4646 [0.5194] | 0.5411 [0.5881] | 0.6176 [0.6567] |
| 33.0000 | 0.3090 [0.3788] | 0.3858 [0.4478] | 0.4626 [0.5169] | 0.5393 [0.5859] | 0.6161 [0.6549] |
| 34.0000 | 0.3065 [0.3757] | 0.3836 [0.4451] | 0.4606 [0.5144] | 0.5377 [0.5838] | 0.6147 [0.6532] |
| 35.0000 | 0.3041 [0.3727] | 0.3814 [0.4424] | 0.4588 [0.5121] | 0.5361 [0.5818] | 0.6134 [0.6515] |
| 36.0000 | 0.3018 [0.3697] | 0.3794 [0.4398] | 0.4569 [0.5098] | 0.5345 [0.5798] | 0.6121 [0.6499] |
| 37.0000 | 0.2995 [0.3669] | 0.3774 [0.4373] | 0.4552 [0.5076] | 0.5330 [0.5780] | 0.6109 [0.6483] |
| 38.0000 | 0.2974 [0.3642] | 0.3755 [0.4349] | 0.4535 [0.5055] | 0.5316 [0.5761] | 0.6097 [0.6468] |
| 39.0000 | 0.2953 [0.3616] | 0.3736 [0.4325] | 0.4519 [0.5034] | 0.5302 [0.5744] | 0.6085 [0.6453] |
| 40.0000 | 0.2933 [0.3590] | 0.3718 [0.4302] | 0.4503 [0.5015] | 0.5289 [0.5727] | 0.6074 [0.6439] |
| 41.0000 | 0.2913 [0.3566] | 0.3701 [0.4280] | 0.4488 [0.4995] | 0.5276 [0.5710] | 0.6063 [0.6425] |
| 42.0000 | 0.2894 [0.3542] | 0.3684 [0.4259] | 0.4473 [0.4977] | 0.5263 [0.5694] | 0.6052 [0.6412] |
| 43.0000 | 0.2876 [0.3518] | 0.3668 [0.4239] | 0.4459 [0.4959] | 0.5251 [0.5679] | 0.6042 [0.6399] |
| 44.0000 | 0.2858 [0.3496] | 0.3652 [0.4218] | 0.4445 [0.4941] | 0.5239 [0.5664] | 0.6032 [0.6387] |
| 45.0000 | 0.2841 [0.3474] | 0.3636 [0.4199] | 0.4432 [0.4924] | 0.5227 [0.5649] | 0.6023 [0.6374] |
| 46.0000 | 0.2824 [0.3453] | 0.3622 [0.4180] | 0.4419 [0.4908] | 0.5216 [0.5635] | 0.6014 [0.6363] |
| 47.0000 | 0.2808 [0.3432] | 0.3607 [0.4162] | 0.4406 [0.4891] | 0.5205 [0.5621] | 0.6004 [0.6351] |
| 48.0000 | 0.2792 [0.3412] | 0.3593 [0.4144] | 0.4394 [0.4876] | 0.5195 [0.5608] | 0.5996 [0.6340] |
| 49.0000 | 0.2777 [0.3392] | 0.3579 [0.4126] | 0.4382 [0.4860] | 0.5185 [0.5595] | 0.5987 [0.6329] |
| 50.0000 | 0.2762 [0.3373] | 0.3566 [0.4109] | 0.4370 [0.4846] | 0.5175 [0.5582] | 0.5979 [0.6318] |

TABLE 7.2. Critical Values ($C_0$) for given $c$ and $r$ values.

| r | c=0.6 | c=0.7 | c=0.8 | c=0.9 |
|---|---|---|---|---|
| 1.0000 | 0.8665 [0.9131] | 0.8999 [0.9349] | 0.9332 [0.9566] | 0.9666 [0.9783] |
| 2.0000 | 0.8314 [0.8795] | 0.8735 [0.9096] | 0.9157 [0.9397] | 0.9578 [0.9699] |
| 3.0000 | 0.8094 [0.8572] | 0.8570 [0.8929] | 0.9047 [0.9286] | 0.9523 [0.9643] |
| 4.0000 | 0.7936 [0.8407] | 0.8452 [0.8805] | 0.8968 [0.9204] | 0.9484 [0.9602] |
| 5.0000 | 0.7815 [0.8277] | 0.8361 [0.8707] | 0.8908 [0.9138] | 0.9454 [0.9569] |
| 6.0000 | 0.7717 [0.8169] | 0.8288 [0.8627] | 0.8859 [0.9085] | 0.9429 [0.9542] |
| 7.0000 | 0.7636 [0.8078] | 0.8227 [0.8559] | 0.8818 [0.9039] | 0.9409 [0.9520] |
| 8.0000 | 0.7566 [0.8000] | 0.8175 [0.8500] | 0.8783 [0.9000] | 0.9392 [0.9500] |
| 9.0000 | 0.7506 [0.7931] | 0.8130 [0.8449] | 0.8753 [0.8966] | 0.9377 [0.9483] |
| 10.0000 | 0.7453 [0.7870] | 0.8090 [0.8403] | 0.8727 [0.8935] | 0.9363 [0.9468] |
| 11.0000 | 0.7406 [0.7816] | 0.8054 [0.8362] | 0.8703 [0.8908] | 0.9351 [0.9454] |
| 12.0000 | 0.7364 [0.7766] | 0.8023 [0.8325] | 0.8682 [0.8883] | 0.9341 [0.9442] |
| 13.0000 | 0.7325 [0.7721] | 0.7994 [0.8291] | 0.8663 [0.8861] | 0.9331 [0.9430] |
| 14.0000 | 0.7291 [0.7680] | 0.7968 [0.8260] | 0.8645 [0.8840] | 0.9323 [0.9420] |
| 15.0000 | 0.7259 [0.7642] | 0.7944 [0.8232] | 0.8629 [0.8821] | 0.9315 [0.9411] |
| 16.0000 | 0.7229 [0.7607] | 0.7922 [0.8205] | 0.8615 [0.8803] | 0.9307 [0.9402] |
| 17.0000 | 0.7202 [0.7574] | 0.7901 [0.8181] | 0.8601 [0.8787] | 0.9300 [0.9394] |
| 18.0000 | 0.7176 [0.7543] | 0.7882 [0.8158] | 0.8588 [0.8772] | 0.9294 [0.9386] |
| 19.0000 | 0.7153 [0.7515] | 0.7865 [0.8136] | 0.8576 [0.8757] | 0.9288 [0.9379] |
| 20.0000 | 0.7130 [0.7488] | 0.7848 [0.8116] | 0.8565 [0.8744] | 0.9283 [0.9372] |
| 21.0000 | 0.7110 [0.7462] | 0.7832 [0.8097] | 0.8555 [0.8731] | 0.9277 [0.9366] |
| 22.0000 | 0.7090 [0.7438] | 0.7817 [0.8079] | 0.8545 [0.8719] | 0.9272 [0.9360] |
| 23.0000 | 0.7071 [0.7416] | 0.7804 [0.8062] | 0.8536 [0.8708] | 0.9268 [0.9354] |
| 24.0000 | 0.7054 [0.7394] | 0.7790 [0.8046] | 0.8527 [0.8697] | 0.9263 [0.9349] |
| 25.0000 | 0.7037 [0.7374] | 0.7778 [0.8030] | 0.8519 [0.8687] | 0.9259 [0.9343] |
| 26.0000 | 0.7021 [0.7354] | 0.7766 [0.8016] | 0.8511 [0.8677] | 0.9255 [0.9339] |
| 27.0000 | 0.7006 [0.7336] | 0.7755 [0.8002] | 0.8503 [0.8668] | 0.9252 [0.9334] |
| 28.0000 | 0.6992 [0.7318] | 0.7744 [0.7988] | 0.8496 [0.8659] | 0.9248 [0.9329] |
| 29.0000 | 0.6978 [0.7301] | 0.7734 [0.7976] | 0.8489 [0.8650] | 0.9245 [0.9325] |
| 30.0000 | 0.6965 [0.7284] | 0.7724 [0.7963] | 0.8483 [0.8642] | 0.9241 [0.9321] |
| 31.0000 | 0.6953 [0.7269] | 0.7714 [0.7952] | 0.8476 [0.8634] | 0.9238 [0.9317] |
| 32.0000 | 0.6941 [0.7254] | 0.7705 [0.7940] | 0.8470 [0.8627] | 0.9235 [0.9313] |
| 33.0000 | 0.6929 [0.7239] | 0.7697 [0.7929] | 0.8464 [0.8620] | 0.9232 [0.9310] |
| 34.0000 | 0.6918 [0.7225] | 0.7688 [0.7919] | 0.8459 [0.8613] | 0.9229 [0.9306] |
| 35.0000 | 0.6907 [0.7212] | 0.7680 [0.7909] | 0.8454 [0.8606] | 0.9227 [0.9303] |
| 36.0000 | 0.6897 [0.7199] | 0.7673 [0.7899] | 0.8448 [0.8599] | 0.9224 [0.9300] |
| 37.0000 | 0.6887 [0.7186] | 0.7665 [0.7890] | 0.8443 [0.8593] | 0.9222 [0.9297] |
| 38.0000 | 0.6877 [0.7174] | 0.7658 [0.7881] | 0.8439 [0.8587] | 0.9219 [0.9294] |
| 39.0000 | 0.6868 [0.7163] | 0.7651 [0.7872] | 0.8434 [0.8581] | 0.9217 [0.9291] |
| 40.0000 | 0.6859 [0.7151] | 0.7644 [0.7863] | 0.8430 [0.8576] | 0.9215 [0.9288] |
| 41.0000 | 0.6850 [0.7140] | 0.7638 [0.7855] | 0.8425 [0.8570] | 0.9213 [0.9285] |
| 42.0000 | 0.6842 [0.7130] | 0.7631 [0.7847] | 0.8421 [0.8565] | 0.9210 [0.9282] |
| 43.0000 | 0.6834 [0.7119] | 0.7625 [0.7839] | 0.8417 [0.8560] | 0.9208 [0.9280] |
| 44.0000 | 0.6826 [0.7109] | 0.7619 [0.7832] | 0.8413 [0.8555] | 0.9206 [0.9277] |
| 45.0000 | 0.6818 [0.7099] | 0.7614 [0.7825] | 0.8409 [0.8550] | 0.9205 [0.9275] |
| 46.0000 | 0.6811 [0.7090] | 0.7608 [0.7818] | 0.8405 [0.8545] | 0.9203 [0.9273] |
| 47.0000 | 0.6804 [0.7081] | 0.7603 [0.7811] | 0.8402 [0.8540] | 0.9201 [0.9270] |
| 48.0000 | 0.6797 [0.7072] | 0.7597 [0.7804] | 0.8398 [0.8536] | 0.9199 [0.9268] |
| 49.0000 | 0.6790 [0.7063] | 0.7592 [0.7797] | 0.8395 [0.8532] | 0.9197 [0.9266] |
| 50.0000 | 0.6783 [0.7055] | 0.7587 [0.7791] | 0.8392 [0.8527] | 0.9196 [0.9264] |

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